AN ANALYTIC APPROACH TO FFAG OPTICS

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1. Thin-lens formulae (review of BNL Workshop talk).

2. Thick-lens formulae.

3. Comparison with lattice-code results.

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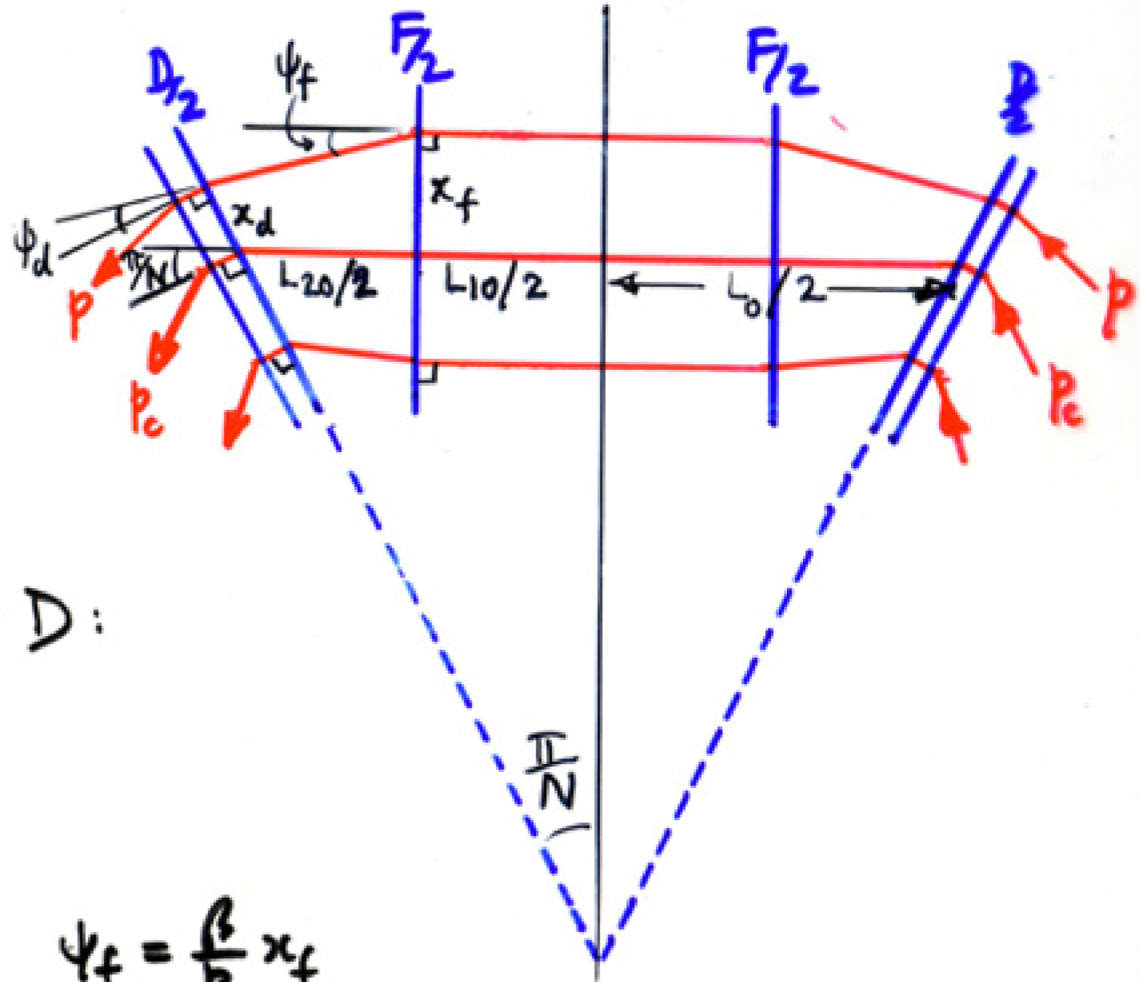
THIN BEND TRIPLET

Assume FDF cells with thin bends:—

The "central orbit" — N-sided polygon — is on axis in F quad for momentum p_c, so all bending occurs at D:

$$e B_c \frac{l_d}{2} = \frac{\pi}{N} p_c .$$



For other momenta p

$$B_f = B_f' x_f \qquad \psi_f = \frac{\beta}{p} x_f$$

$$B_d = B_c - B_d' x_d \qquad \psi_d = \frac{\pi}{N}\frac{p_c}{p} - \frac{\beta}{p} x_d$$

where $\beta \equiv e B_f' l_f/2 = e B_d' l_d/2$ i.e. equal F and D quad strengths.

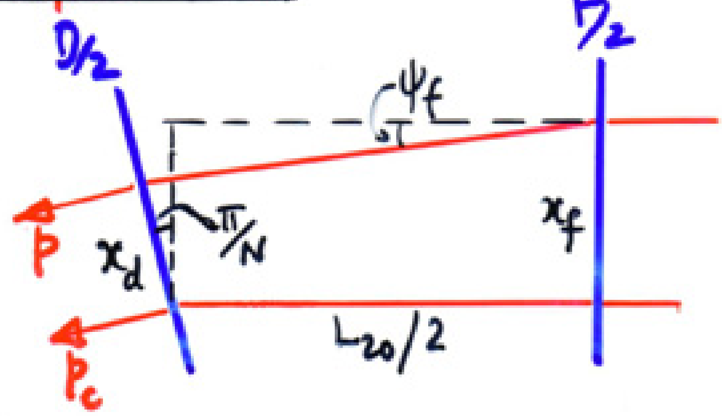
N.B. As $\frac{L_{20}}{2} \rightarrow \frac{L_0}{2}$ so FDF $\longrightarrow$ FODO.

OFFSETS AT THE QUADS

Total bend $\psi_f + \psi_d = \pi/N$

$\therefore$

$$x_f - x_d = \frac{\pi}{N\beta}(p - p_c)$$



But for $\psi_f \to 0$, $\pi/N \to 0$

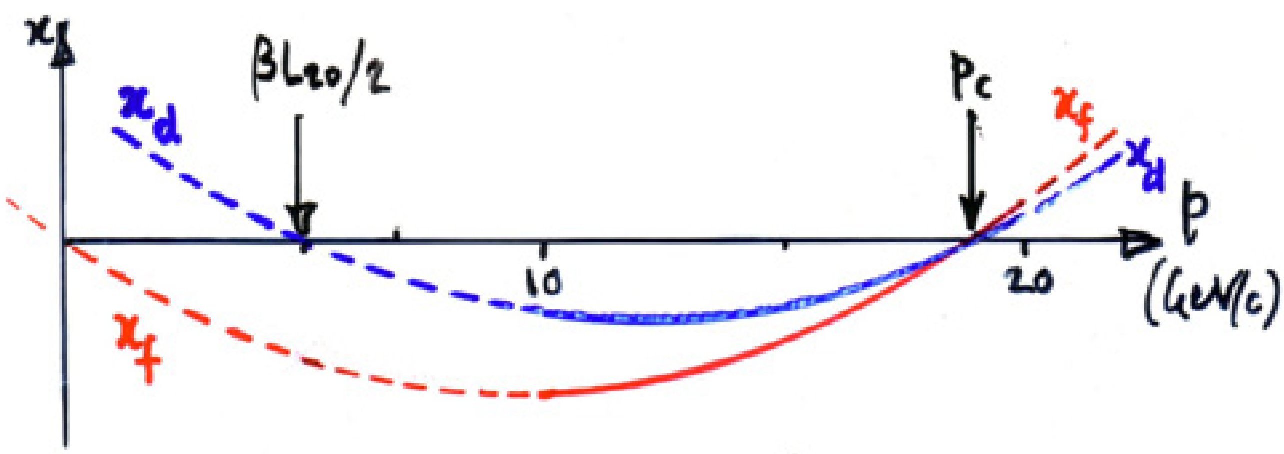
$$x_f - x_d \simeq \tfrac{1}{2} L_{20} \psi_f$$

$$= \frac{\beta L_{20}}{2p} x_f$$

$\therefore$

$$x_f = \frac{2\pi}{\beta^2 N L_{20}} p(p - p_c)$$

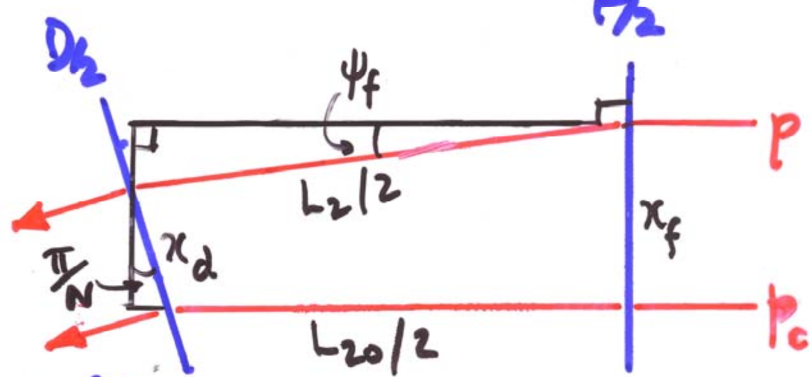
$$x_d = \frac{2\pi}{\beta^2 N L_{20}} (p - p_c)\left(p - \frac{\beta L_{20}}{2}\right)$$



N.B. x amplitude factor $\propto 1/L_{20}$

– so for the same β and N $\quad \dfrac{x\,(\text{FDF})}{x\,(\text{FODO})} = \dfrac{L_{20}\,(\text{FODO})}{L_{20}\,(\text{FDF})} \simeq 3$

PATH LENGTH



$$\frac{L_2}{2}\cos\psi_f = \frac{L_{20}}{2} + x_d \sin\frac{\pi}{N}$$

$$\therefore\; L_2 \simeq \left(L_{20} + \frac{2\pi}{N}x_d\right)\left(1+\frac{1}{2}\beta^2\frac{x_f^2}{p^2}\right)$$

$$\&\; \Delta L \equiv L_2 - L_{20}$$

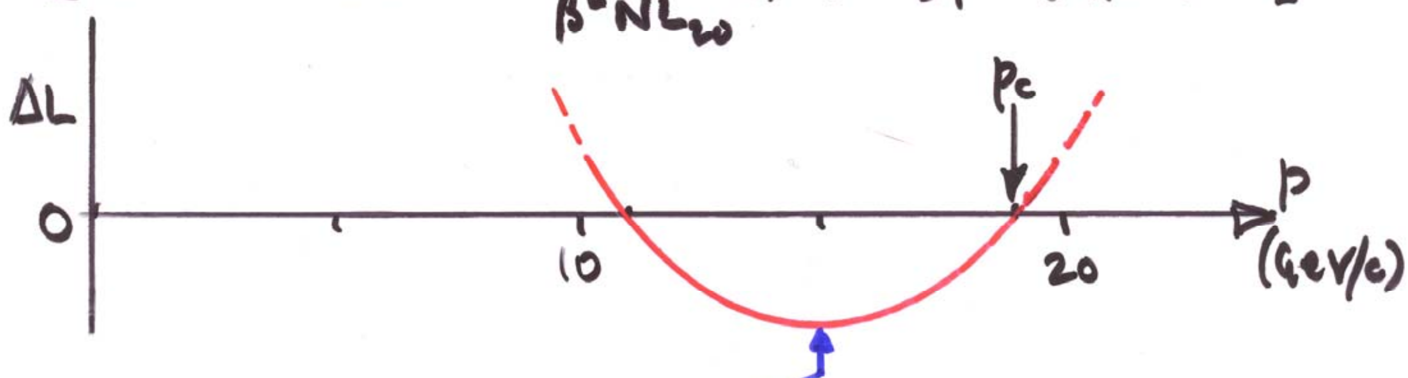
$$\simeq \frac{2\pi}{N}x_d + \frac{1}{2}\beta^2 L_{20}\left(\frac{x_f}{p}\right)^2 \quad \text{for } x_d \ll NL_{20},\ \psi_f \to 0$$

i.e.

$$\boxed{\Delta L = \frac{6\pi^2}{\beta^2 N^2 L_{20}}(p-p_c)\left[p - \frac{1}{3}(p_c+\beta L_{20})\right]}$$

Over the whole circumference

$$\Delta C = N\Delta L = \frac{6\pi^2}{\beta^2 N L_{20}}(p-p_c)\left[p-\frac{1}{3}(p_c+\beta L_{20})\right]$$



Minimum L occurs for $p_{\check{L}} = \frac{1}{3}\left(2p_c + \frac{\beta L_{20}}{2}\right)$

– or inversely $p_c = \frac{1}{2}\left(p_{\check{L}} - \frac{\beta L_{20}}{2}\right)$

– so for $p_{\check{L}} = 15$ GeV/c, $\frac{\beta L_{20}}{2} = 7$ GeV/c $\rightarrow p_c = 19$ GeV/c.

N.B. $\Delta C \propto 1/\beta^2 N L_{20}$ – just like x_f, x_d.

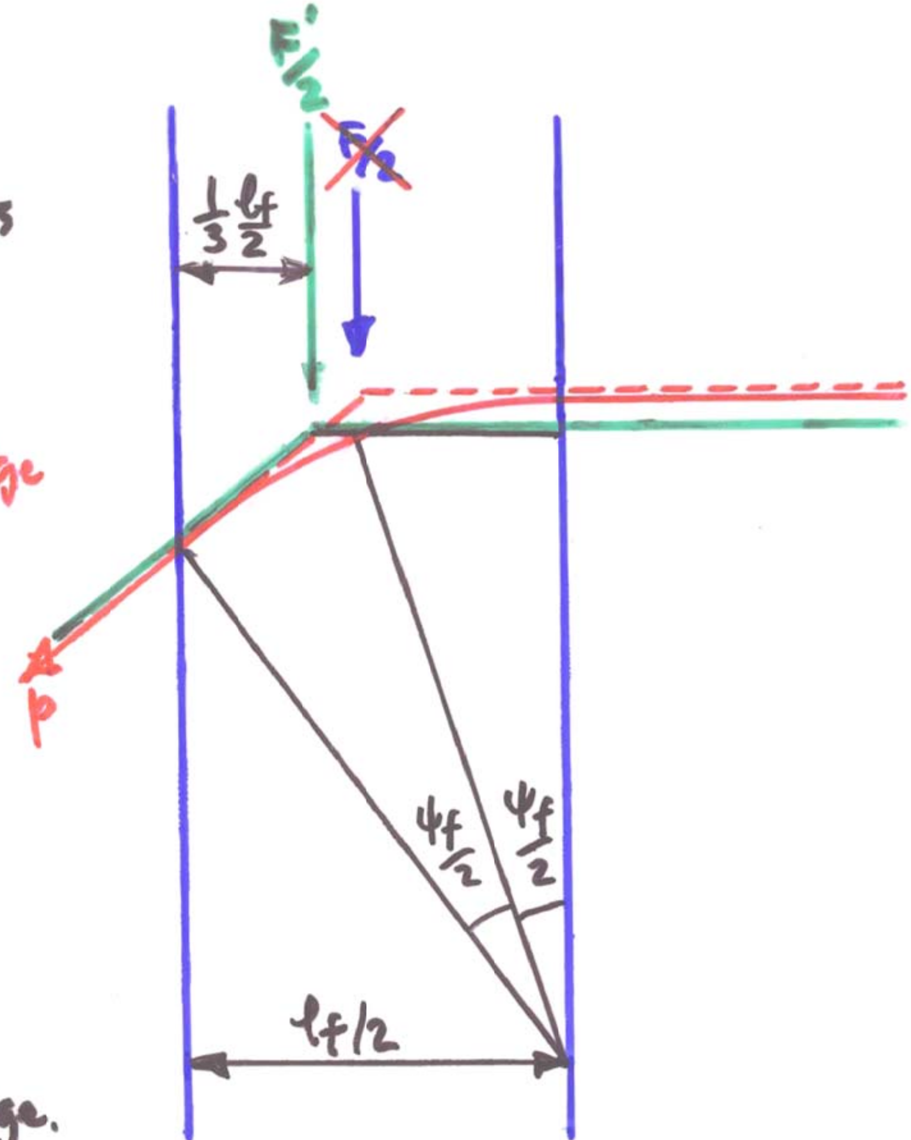
THICK BENDS

Orbit curvature within the magnets makes

$$\hat{x}_{thick} \neq \hat{x}_{thin}$$

– so placing thin bend at hi-x edge of a thick ½-bend → B error.

∴ shift the thin bend so that $B(F'/2) = \langle B(arc) \rangle$.



For ‖-ended magnets place $\frac{F'}{2}$ $\frac{1}{3}\frac{l_f}{2}$ from D-facing edge.

This reduces the D-F spacing L_{20} to

$$L'_{20} = l_{fd} + \frac{1}{3}\left(\frac{l_f}{2} + \frac{l_d}{2}\right)$$

– a significant correction to $L_{20} = l_{fd} + \frac{l_f}{4} + \frac{l_d}{2}$ when $l_{fd} \simeq 20$ cm

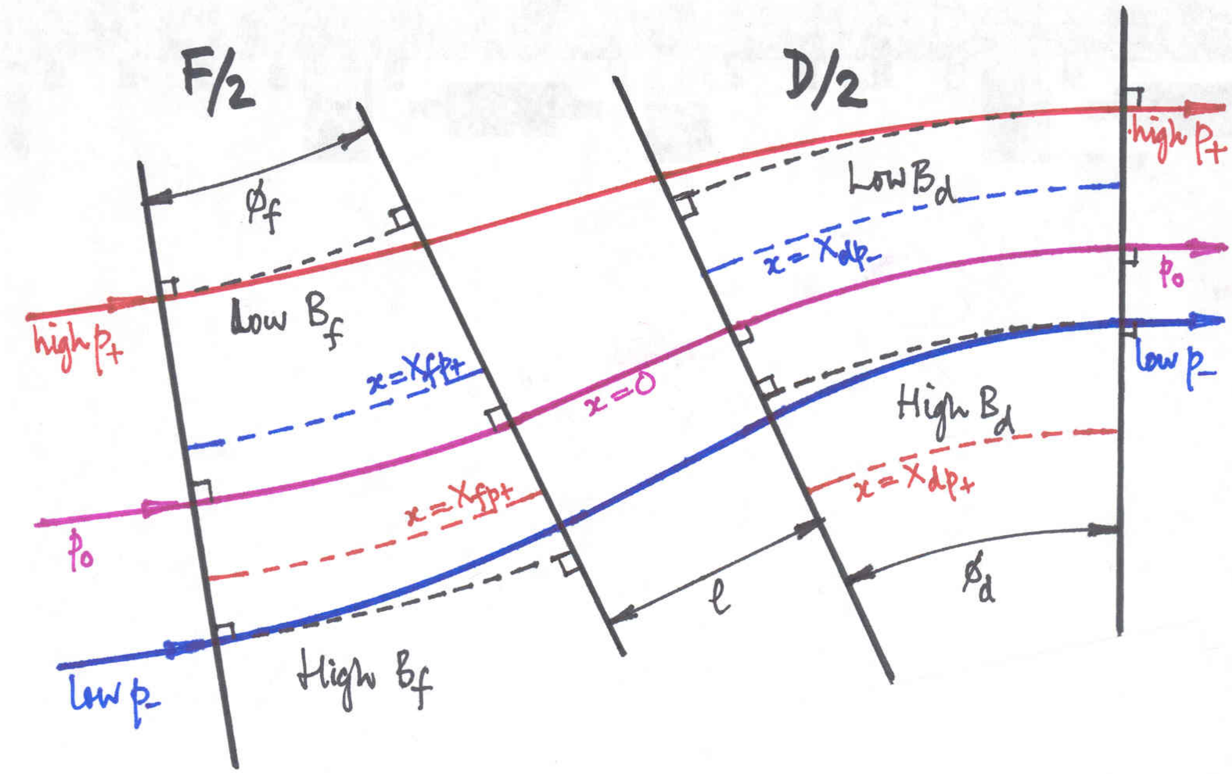
For sector magnets, $L'_{20} \simeq l_{fd} + \frac{1}{3}\left(\frac{l_f}{2} + \frac{l_d}{2}\right)$

		Dejan			Scott					Carol
		FDF Triplets			FDF	Doublet	F0D0-1	F0D0-2	F0D0PAC	F0D0
Circumference (m)	**C**	**323**	**328**	**348**	**481.74**	**436.53**	**612.18**	**470.00**	**1200**	**2041**
Number of cells	**N**	66	72	80	93	101	113	82	200	314
Cell length (m)	**$C/N = L_0$**	4.89	4.56	4.35	5.18	4.32	5.42	5.73	6.00	6.5
D-bend length (m)	**$l_d = BL$**	**1.50**	**1.00**	**0.80**	**1.28**	**1.01**	**0.81**	**0.99**	**1.00**	**0.35**
F-quad length (m)	**$l_f/2 = QLF$**	0.50	0.58	0.58	0.45	0.41	0.30	0.37	0.50	0.075
F-D spacing (m)	**D3P = dl3**	0.17	0.20	0.20	0.50	0.50	2.00	2.00	2.00	3.00
Effective F-D spacing / L_0	**L'_{20} / L_0**	0.24	0.25	0.24	0.33	0.26	0.83	0.80	0.78	0.95
F gradient (T/m)	**$B'_f = GF1$**	67.42	62.43	66.74	50.1	43.09	39.42	30.64	20.38	75.90
D gradient (T/m)	**$B'_d = -GDD2$**	35.57	56.65	75.30	30.4	35.68	29.97	23.70	20.42	32.45
Quad strength factor (mean)	β	9.05	9.67	10.32	6.30	5.32	3.61	3.46	3.06	1.70
Time-of-flight range/cavity (ps)	**$DeltaC/(c*n_{cav})$**	**11.9**	**9.0**	**6.9**	**8.3**	**7.9**	**6.6**	**6.7**	**2.8**	**5.5**
Path length variation (cm)	**DeltaC**	**23.5**	**19.5**	**16.5**	**23.3**	**24.0**	**22.5**	**32.9**	**17.0**	**51.6**
" " (tracked)		**14.2**	**11.7**	**9.9**	**17.1**	**21.4**	**24.0**	**35.0**	**18**	**51**
D-bend field (T) @ 15 GeV/c	**B_d = by0**	5.08	6.69	7.49	4.53	4.89	4.85	5.61	2.34	
F-quad field (T) @ 15 GeV/c	**B_f = byq**	-2.85	-2.00	-1.78	-2.68	-2.23	-1.93	-2.30	-0.77	
Momentum (GeV/c) for $B_f = 0$	**p_c**		19.1	19.0	19.8	20.4	18.5	18.5	18.9	16.6
Momentum @ $x_d = x_c$	**$Beta*L'_{20}/2$**	5.3	5.5	5.4	5.4	4.3	8.1	7.9	7.1	5.3
Momentum @ minimum path	**$p(C_{min})$**		**14.5**	**14.5**						**12.8**
" " (tracked)			**14.5**	**14.5**	**15**	**15**	**15**	**15**	**15**	**12.7**
										6 GeV/c
x offset (cm) in F @ 10 GeV/c	**$x\_f_{10}-x\_f_{15}$**		**-2.1**	**-1.8**	**-2.2**	**-2.6**	**-3.0**	**-4.4**	**-2.1**	**-7.0**
" " (tracked)	**XCOf**	**-2.9**	**-2.5**	**-2.1**	**-2.5**	**-2.8**	**-3.3**	**-4.8**	**-2.5**	**-7.0**
x offset (cm) in F @ 20 GeV/c	**$x\_f_{20}-x\_f_{15}$**		**6.4**	**5.4**	**7.2**	**7.7**	**7.8**	**11.3**	**5.7**	**7.8**
" " (tracked)	**XCOf**	**4.6**	**3.9**	**3.3**	**5.3**	**7.4**	**8.4**	**12.3**	**6.1**	**7.7**

N.B. Data computed with thick bend correction factor 1/3 (in place of 1/2).

ORBITS IN FDF SECTOR GRADIENT MAGNETS

The figure shows a half cell:



Each magnet has an equilibrium orbit for each momentum p

- a circular arc (radius ρ_{fp}, ρ_{dp}) with constant field B_{fp}, B_{dp}
- offset by X_{fp}, X_{dp} w.r.t. the reference orbit $x = 0$:

$$X_p = \frac{r_0}{2n_0}\left\{(1-n_0)+\sqrt{(1-n_0)^2 - 4n_0(\Delta p / p_0)}\right\}$$

where $$n_0 = -\frac{r}{B}\frac{dB}{dr}\bigg|_{p_0}$$

For a whole cell, the orbit consists of:

- long and short drifts;
- betatron oscillations
  - sinusoidal in F of amplitude $A_f = x_{f\perp} - X_{fp}$ about X_{fp},
  - hyperbolic in D of amplitude $A_d = x_{d\perp} - X_{dp}$ about X_{dp}.
  - where $x_{f\perp}$ and $x_{d\perp}$ are the offsets at the normal-crossing edge.

MATCHING CONDITIONS

At the ends of the short drift between F and D,
the betatron displacements and divergences are:

$$x_{f/} - X_{fp} = A_f \cos\alpha_f \qquad x_{d/} - X_{dp} = A_d \cosh\alpha_d$$

$$\tan\chi_f \cong \frac{A_f}{\rho_{f0}}\sqrt{1-n_f}\sin\alpha_f \qquad \tan\chi_d \cong \frac{A_d}{\rho_{d0}}\sqrt{n_d-1}\sinh\alpha_d$$

where $\alpha_f \equiv \sqrt{1-n_f}\,\phi_f \qquad \alpha_d \equiv \sqrt{n_d-1}\,\phi_d \qquad \tan\chi = \frac{1}{r}\frac{dr}{d\phi}$

and n_f, n_d are evaluated at X_{fp}, X_{dp}.

Matching the divergences, we find $$\frac{A_f}{A_d} \cong \sqrt{\frac{n_d-1}{1-n_f}}\,\frac{\rho_{f0}\sinh\alpha_d}{\rho_{d0}\sin\alpha_f}.$$

Matching the displacements, so that: $x_{f/} - x_{d/} = \ell\tan\chi_d$

we get:

$$A_d = \frac{X_{fp} - X_{dp}}{\cosh\alpha_d + \frac{\sqrt{n_d-1}}{\rho_{d0}}\sinh\alpha_d\left[\ell - \frac{\rho_{f0}}{\sqrt{1-n_f}\tan\alpha_f}\right]}$$

and hence A_f too, enabling us to:

- compute the offsets $x_p(\phi)$ for any azimuthal angle ϕ
- but giving no insight into the momentum dependence,
  - since X_{fp}, X_{dp}, n_f, n_d, α_f, and α_d all individually vary with p.
- though Shane has gone much further, using Mathematica to extract the p-dependence, β and ν values........

ORBIT LENGTH

Integrating along the various orbit segments (F, D, short drift), we find the following deviations in path length between momenta p and p_0 (the reference orbit):

$$\Delta s_f \cong -\left(X_{fp} + A_f\right)\phi_f + \frac{A_f}{\sqrt{1-n_f}}(\alpha - \sin\alpha) + \frac{\sqrt{1-n_f}\,A_f^2}{8\rho_{f0}}(2\alpha - \sin 2\alpha)$$

$$\Delta s_d \cong \left(X_{dp} + A_d\right)\phi_d + \frac{A_d}{\sqrt{n_d - 1}}(\sinh\alpha - \alpha) + \frac{\sqrt{n_d - 1}A_d^2}{8\rho_{d0}}(\sinh 2\alpha - 2\alpha)$$

$$\Delta s_\ell = \ell\left(\sec\chi_{fd} - 1\right)$$

Terms of higher order in A_f/ρ_{f0} and A_d/ρ_{d0} are negligible and have been discarded.

The formulae for offsets x and path length variation Δs give results in excellent agreement with lattice codes.

	Dejan			Scott		
	FDF Triplets			FDF	F0D0-1	F0D0-2
Circumference C (m)	323	328	348	481.7	612.2	470
Number of cells N	66	72	80	93	113	82
Cell length L_0 = C/N (m)	4.89	4.56	4.35	5.18	5.42	5.73
D-bend length (m)	**1.5**	**1.0**	**0.8**	**1.28**	**0.81**	**0.99**
F-quad length (m)	0.5	0.58	0.58	0.45	0.30	0.37
F-D spacing (m)	0.17	0.20	0.20	0.50	2.00	2.00
D field B_d (T) @ 15 GeV	5.08	6.69	7.49	4.53	4.85	5.61
F field B_f (T) " " " "	-2.85	-2.00	-1.78	-2.68	-1.93	-2.30
D gradient B_d' (T/m)	35.57	56.65	75.30	30.43	29.97	23.70
F gradient B_f' (T/m)	67.42	62.43	66.74	50.11	39.42	30.64
D Curvature ρ_d (m)	9.86	7.48	6.68	11.05	10.32	8.92
F " " " ρ_f (m)	17.56	25.02	28.11	18.64	25.99	21.73
D sector angle φ_d (rad)	0.076	0.067	0.060	0.058	0.039	0.055
F " " " " " φ_f (rad)	0.028	0.023	0.021	0.024	0.012	0.017
D field index n_d-1	68.1	62.4	66.1	73.3	62.8	36.7
F field index 1-n_f	416.3	782.0	1,055.0	348.9	533.1	290.2
$\alpha_d = \varphi_d\sqrt{(n_d-1)}$ (rad)	0.628	0.528	0.487	0.496	0.312	0.336
$\alpha_f = \varphi_f\sqrt{(1-n_f)}$ (rad)	0.581	0.648	0.670	0.451	0.268	0.291

10 GeV/c	Dejan FDF Triplets			Scott FDF	Scott F0D0-1	Scott F0D0-2
Circumference C (m)	323	328	348	481.7	612.2	470
E.O. offset X_{d10} (m) for D	0.0480	0.0398	0.0335	0.0500	0.0545	0.0803
Osc'n amplitude A_d (m)	0.0458	0.0387	0.0330	0.0474	0.0470	0.0694
Offset $x_{d\perp}$ (m) - formula	0.0022	0.0011	0.0005	0.0026	0.0075	0.0109
" " " " " - lattice code	0.0024	0.0010	0.0005	0.0031	0.0077	0.0116
" " " " " - difference	-0.0002	0.0000	0.0000	-0.0004	-0.0003	-0.0007
E.O. offset X_{f10} (m) in F	0.0141	0.0107	0.0089	0.0178	0.0163	0.0250
Osc'n amplitude A_f (m)	0.0436	0.0361	0.0306	0.0459	0.0503	0.0751
Offset $x_{f\perp}$ (m) - formula	-0.0295	-0.0255	-0.0217	-0.0281	-0.0340	-0.0501
" " " " " - lattice code	-0.0290	-0.0250	-0.0213	-0.0270	-0.0331	-0.0478
" " " " " - difference	-0.0006	-0.0005	-0.0004	-0.0010	-0.0009	-0.0024
Extra path ΔC (m) - thin lens	0.2393	0.1974	0.1675	0.2340	0.2247	0.3293
" " " " " " " - formula	0.1339	0.1050	0.0873	0.1793	0.2197	0.3303
" " " " " " - lattice code	0.1434	0.1165	0.0992	0.1943	0.2382	0.3442
" " " " " " - difference	-0.0095	-0.0115	-0.0119	-0.015	-0.0184	-0.0139
20 GeV/c						
E.O. offset X_{d20} (m) for D	-0.0485	-0.0402	-0.0338	-0.0505	-0.0551	-0.0818
Osc'n amplitude A_d (m)	0.0713	0.0598	0.0507	0.0776	0.0942	0.1359
Offset $x_{d\perp}$ (m) - formula	0.0228	0.0196	0.0168	0.0271	0.0391	0.0541
" " " " " - lattice code	0.0232	0.0197	0.0169	0.0292	0.0422	0.0621
" " " " " - difference	-0.0004	-0.0002	-0.0001	-0.0021	-0.0032	-0.0080
E.O. offset X_{f20} (m) in F	-0.0140	-0.0107	-0.0089	-0.0178	-0.0162	-0.0249
Osc'n amplitude A_f (m)	0.0597	0.0493	0.0416	0.0686	0.0950	0.1347
Offset $x_{f\perp}$ (m) - formula	0.0456	0.0386	0.0327	0.0508	0.0788	0.1098
" " " " " - lattice code	0.0465	0.0388	0.0328	0.0539	0.0839	0.1227
" " " " " - difference	-0.0008	-0.0002	-0.0001	-0.0031	-0.0051	-0.0129
Extra path ΔC (m) - thin lens	0.2393	0.1974	0.1675	0.2340	0.2247	0.3293
" " " " " " " - formula	0.1360	0.1154	0.0981	0.1516	0.2464	0.3555
" " " " " " - lattice code	0.1409	0.1184	0.0994	0.1472	0.2412	0.3540
" " " " " " - difference	-0.0049	-0.0030	-0.0014	0.0044	0.0052	0.0015

CONCLUSIONS

The thick-lens formulae:

- are in excellent agreement with the results from lattice codes;

- provide further confirmation that the lattice-code results for path-length variation are correct;

- are accurate enough for estimating at least some orbit parameters:
  - and so can be a useful guide to choosing FFAG design parameters (as Sandro and Shane have shown much more thoroughly).