Control of the Amplitude of the Dispersion Function

The radial offset dependence on momentum is defined as:

$\Delta x = D_x \, dp/p,$

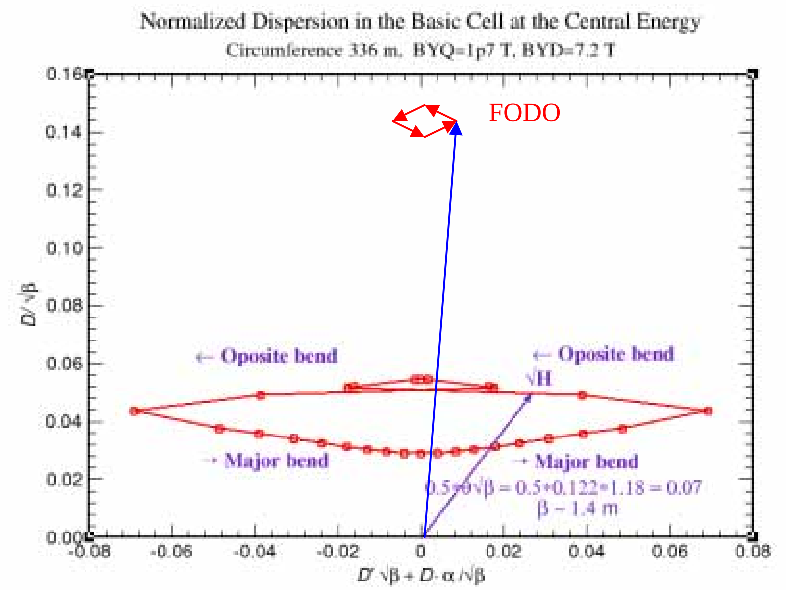
The particle path around the ring is defined as:

$$\Delta C = \left[\oint_C \frac{D(s)}{\rho} \, ds \right] \delta \quad where \ \delta = \frac{\Delta p}{p}.$$

$$D'' + \left(K_x(s) - \frac{2}{\rho^2} + \delta \, K(s) + O(\delta^2) \right) D = \frac{1}{\rho} + O(\delta)$$

$$D'' + K_x D = \frac{1}{\rho}, \qquad K_x = \frac{1}{\rho^2} + K(s)$$

$$\xi = \frac{D}{\sqrt{\beta}} \quad and \quad \chi = D'\sqrt{\beta} + \frac{\alpha D}{\sqrt{\beta}}$$

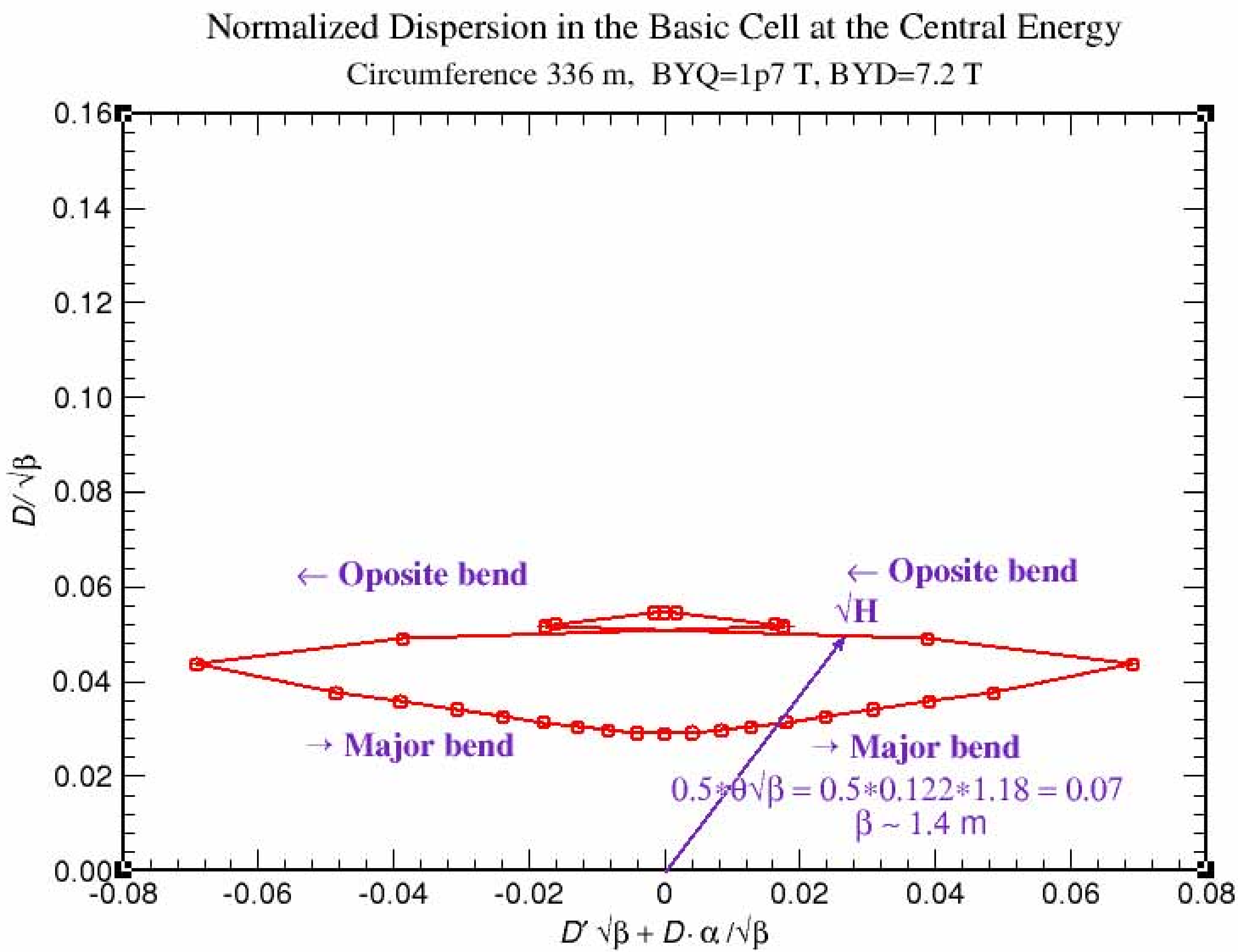


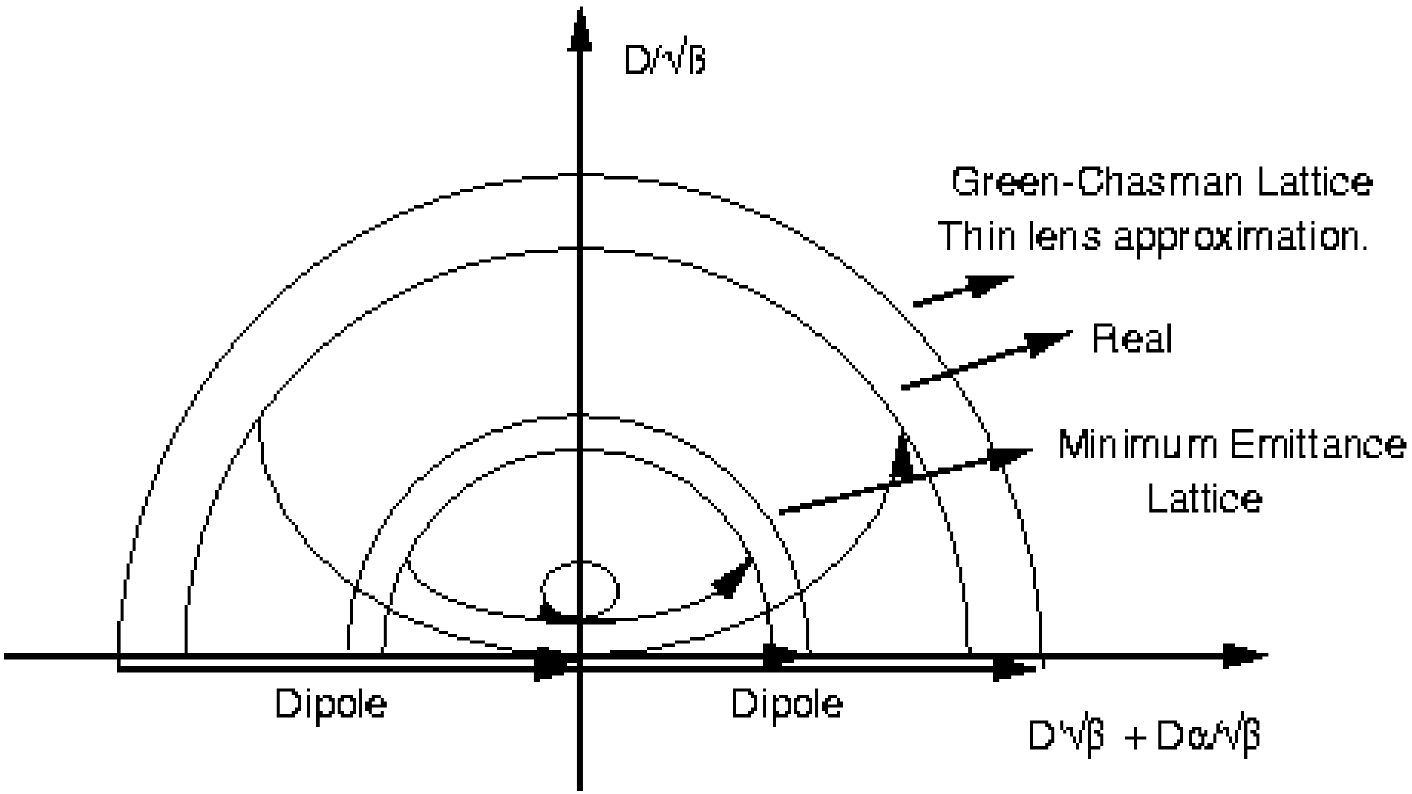
$$H(D, D') \equiv \left(\frac{D}{\sqrt{\beta}}\right)^2 + \left(D'\sqrt{\beta} + \frac{\alpha D}{\sqrt{\beta}}\right)^2 ,$$

The amplitude value of '***H***' is determined within the bending element.
It is enough to minimize '***H***' within the dipole to obtain the minimum.
The average value of the '***H***' function is always proportional to the :
$(\rho\theta)^3$ and a merit factor ***F*** in minimization had been previously introduced as:

$$<H> \approx F\ \rho\,\theta^3 \qquad \frac{d}{dD_o}<H> = 0, \quad \frac{d}{dD'_o}<H> = 0\ , \ \frac{d}{d\beta_o}<H> = 0$$

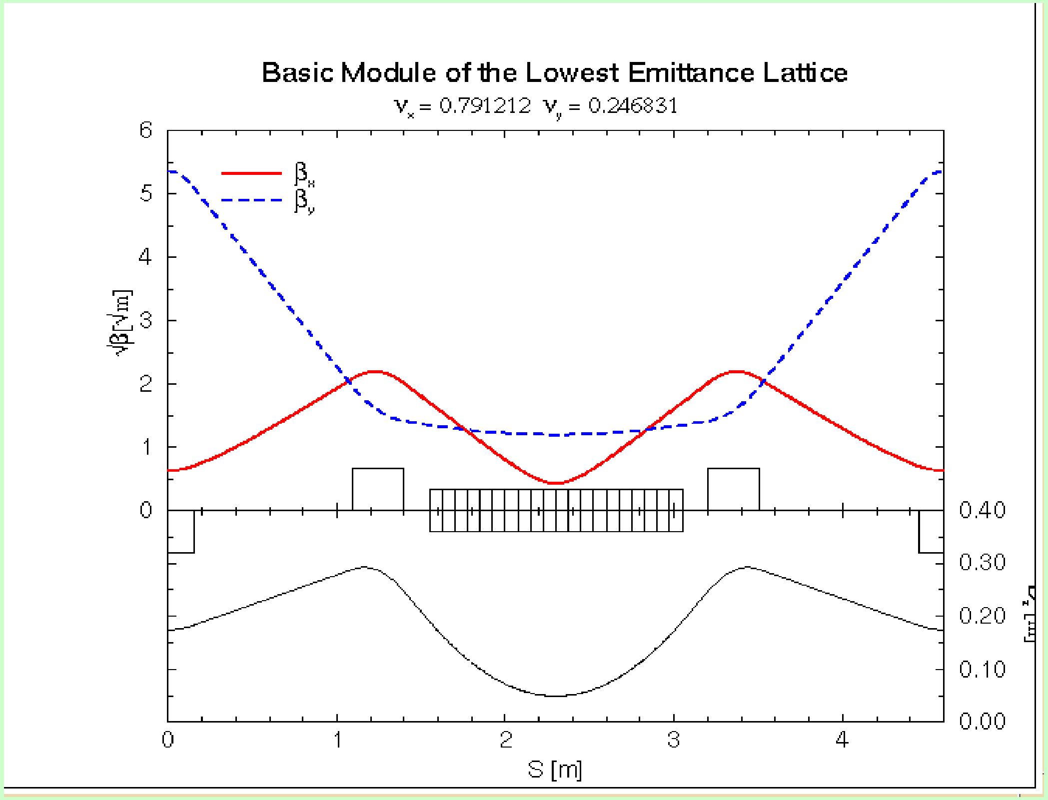
The major conclusions from previously compiled analytical formulas for different lattices like FODO, doublet, triplet, double bend achromat, triple bend achromat etc. comparisons are listed bellow:


Normalized Dispersion in the Basic Cell at the Central Energy
Circumference 336 m, BYQ=1p7 T, BYD=7.2 T
D/ √β
D′ √β + D· α /√β
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08
← Oposite bend
← Oposite bend
√H
→ Major bend
→ Major bend
0.5*θ√β = 0.5*0.122*1.18 = 0.07
β ~ 1.4 m


D/√β
Green-Chasman Lattice
Thin lens approximation.
Real
Minimum Emittance
Lattice
Dipole
Dipole
D'√β + Dα/√β

The minimum emittance lattice:

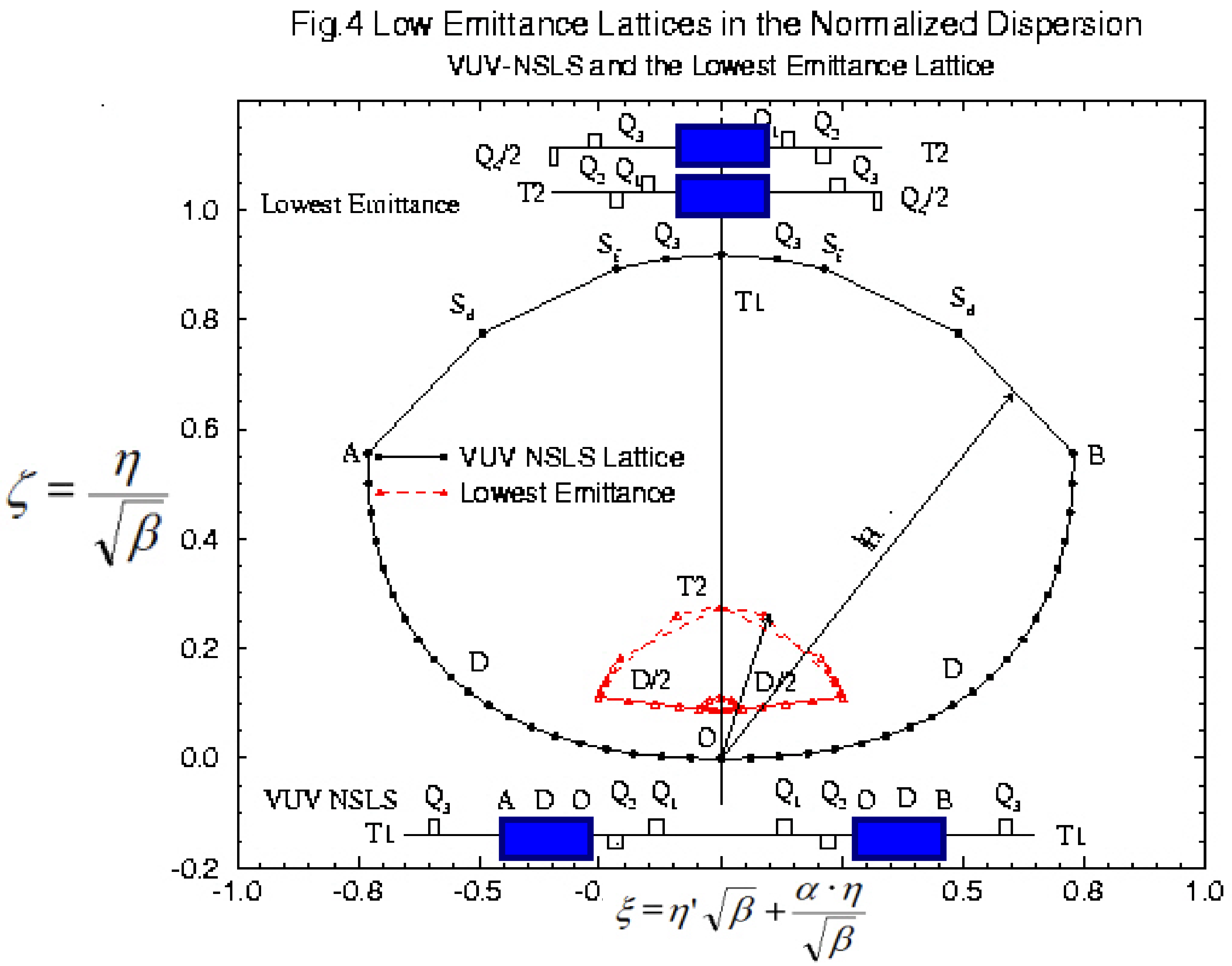
- The minimum emittance lattice requires reduction of the function H:
  - The normalized dispersion amplitude corresponds to the $\langle H \rangle^{1/2}$
  - Conditions are for the minimum of the betatron function β_x and dispersion function D_x to have small values at the middle of the dipole (combined function dipole makes it even smaller).



$$\beta_{min} = Ld/2\sqrt{15}$$

$$D_{xmin} = \theta * Ld/24$$

NSLS reduction of the emittance: 10 times



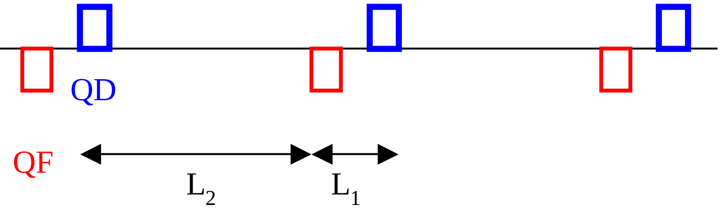
The major conclusions from previously compiled analytical formulas for different lattices like FODO, doublet, triplet, double bend achromat, triple bend achromat etc. comparisons are listed bellow:
The FODO cell *H* function and conditions for the minimum of the average value < H > through the cell are:
The dispersion function in the FODO cell:

$$H_{FODO} = L\theta^2 \frac{\cos\left(\frac{\phi}{2}\right)}{\sin^3\left(\frac{\phi}{2}\right)\left(1+\sin\left(\frac{\phi}{2}\right)\right)}\left(1+\frac{1}{2}\sin\left(\frac{\phi}{2}\right)\right)^2, \quad where\ \frac{L}{2f}=\sin\left(\frac{\phi}{2}\right),$$

$$F_{FODO} \approx \frac{1-\frac{3}{4}\sin^2\left(\frac{\phi}{2}\right)}{\sin^3\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)}$$

a minimum at $\phi \sim 140^o$

Doublets:



$$\sin\left(\frac{\phi}{2}\right)=\frac{\sqrt{L_1L_2}}{2f}$$

The merit factor for different lattices [Accelerator Physics, 1999, S.Y. Lee]:

$$F_{TRIPLET}=\frac{1}{12\sqrt{15}}\approx 0.0215$$

$$F_{DoubleBendAchromat}=\frac{1}{4\sqrt{15}}\approx 0.06455=3*F_{TRIPLET}$$

$$F_{FODO}\approx 1.3\approx 60*F_{TRIPLET}$$

The dispersion function in the combined function dipole:

$$D' = \left(D_o \sqrt{K} + \frac{1}{\rho\sqrt{K}} \right) \sinh\phi + D_o^{'} \cosh\phi$$

$$D = \frac{1}{\rho K}\left(\cosh\phi - 1\right) + D_o \cosh\phi + \frac{1}{\sqrt{K}} D_o^{'} \sinh\phi$$

$$\phi = \sqrt{K}\, s, \quad K = -\frac{\frac{\partial B}{\partial x}}{B\rho} - \frac{1}{\rho^2} \qquad D_o = D(s=0), \quad D_o^{'} = D'(s=0)$$

$$H(\phi) = H_o + \frac{2}{\rho}\left(\alpha_o D_o + \beta_o D_o^{'}\right)\sinh\phi - \frac{2}{\rho K}\left(\gamma_o D_o + \alpha_o D_o^{'}\right)\left(\cosh\phi - 1\right) +$$

$$\frac{\beta_o}{\rho^2 K^{3/2}} \sinh\phi\left(\cosh\phi - 1\right)^2 - \frac{2\alpha_o}{\rho^2 K^{3/2}} \sinh\phi\left(\cosh\phi - 1\right)$$

$$H_o = \gamma_o D_o^2 + 2\alpha_o D_o D_o^{'} + \beta_o D_o^{'\,2}$$

$$< H > = H_o + \rho\theta^3 \left[\left(\alpha_o \frac{D_o}{L\theta} + \frac{\beta_o}{L} \frac{D_o^{'}}{\theta} \right) E(q) - \frac{1}{3} \left(\gamma_o L_o + \alpha_o \frac{D_o^{'}}{\theta} \right) F(q) + \frac{\beta_o}{3L} A(q) - \frac{\alpha_o}{4} B(q) + \frac{\gamma_o L}{20} C(q) \right]$$

$$\frac{\partial < H >}{\partial D_o} = 0, \qquad \frac{\partial < H >}{\partial D_o^{'}} = 0$$

$$D_o = L\theta \frac{F(q)}{6}, \qquad D_o^{'} = -\frac{1}{2} E(q)$$

$$< H >_{\min} = \frac{\tilde{G}}{12\sqrt{15}} \rho\theta^3 , \qquad \tilde{G} = \sqrt{16\tilde{A}\tilde{C} - 15\tilde{B}^2} ,$$

$$\tilde{A} = 4A - 3E^2, \quad \tilde{B} = 3B - 2EF, \quad \tilde{C} = \frac{9}{4} C - \frac{5}{4} F^2$$

$$E(q)=\frac{2(\cosh q-1)}{q^2}$$

$$F(q)=\frac{6(\sinh q-q)}{q^3}$$

$$A(q)=\frac{3\sinh 2q-6q}{4q^3}$$

$$B(q)=\frac{6-8\cosh 2q+2\cosh 2q}{q^4}$$

$$C(q)=\frac{30q-40\sinh+5\sinh 2q}{q^5}$$