Fast Crossing of Betatron Resonances

Rick Baartman

May 6, 2004

Abstract

The theory of fast crossing is developed, and some applications are given.

Hamiltonian

"Smoothed" Hamiltonian with n harmonic, m order driving term

$$H = \frac{p^2}{2} + \frac{Q^2 x^2}{2} + n\frac{b_{n,m} x^m}{m} \cos(n\theta + \phi)$$

Independent variable is θ, the azimuth around the ring. We let the tune Q be a (slow) function of θ. We are interested in the resonance at $mQ = n$. Equation of motion: $x'' + Q^2 x = -nb_{n,m} x^{m-1} \cos(n\theta + \phi)$. $b_{n,m}$is a strength parameter: $nb_{n,m+1} = \frac{\overline{R}}{\overline{B}} \frac{1}{m!} \frac{\partial^m B_n}{\partial x^m}$

Action-angle variables (J, ψ):

$$x = \sqrt{2J/Q} \cos\psi, \; p = \sqrt{2JQ} \sin\psi \tag{1}$$

New Hamiltonian:

$$H = QJ + \frac{nb_{n,m}}{m} \left(\frac{2J}{Q}\right)^{m/2} \cos^m\psi \cos(n\theta + \phi)$$

Solution

$$\psi' = \frac{\partial H}{\partial J} = Q + \text{ oscillatory term}$$

$$\psi \approx \int Q d\theta = Q_0\theta + Q'\theta^2/2$$

where $Q(\theta) = Q_0 + Q'\theta$, we care about the resonance at $Q_0 = n/m$.

$$J' = nb_{n,m}\left(\frac{2J}{Q_0}\right)^{m/2}\cos^{m-1}\psi\sin\psi\cos(n\theta+\phi)$$

$$J' = nb_{n,m}\left(\frac{2J}{Q_0}\right)^{m/2}\times\ldots$$

$$\left[\frac{1}{2^m}\sin(m\psi - n\theta - \phi) + \text{ other terms}\right]$$

We retain the designated term because it is the only one that does not vary rapidly near resonance; the other terms vary too rapidly to make a net contribution. $m\psi - n\theta = m[Q_0\theta + Q'\theta^2/2] - n\theta = mQ'\theta^2/2$

The action equation is a little simpler if we revert to $A = \sqrt{2J/Q_0}$, the betatron amplitude:

$$\frac{A'}{A^{m-1}} = \frac{nb_{n,m}}{2^m Q_0}\,\sin(mQ'\theta^2/2 - \phi)$$

We see that we get a Fresnel integral. The largest amplitude gain occurs for phase $\phi = \pi/4$:

$$\frac{\Delta(A^{2-m})}{2-m} = \frac{nb_{n,m}}{2^m Q_0}\sqrt{\frac{2\pi}{mQ'}}$$

Note $Q_0 = n/m$. Also, we prefer the tune change per turn, $Q_\tau \equiv 2\pi Q'$,

$$\frac{\Delta(A^{2-m})}{2-m} = \frac{\pi}{2^{m-1}}\,b_{n,m}\sqrt{\frac{m}{Q_\tau}}$$

Of course this does not hold for $m = 2$; in that case, the LHS is $\Delta(\log A)$.

Guignard's formula

Our formula agrees with Guignard's [CERN yellow report 77-10, section 10.7], except that he would have $\pi^{m/2}$ in place of our π (note that Guignard's E is actually $\pi\epsilon$). From this we can generalize the definition of $b_{n,m}$ to non-smooth focusing. Since in the non-smooth case, A is ambiguous, we revert to expression in terms of the particle's emittance ϵ.

$$\left|\frac{\Delta(\epsilon^{1-m/2})}{m-2}\right| = \frac{\pi}{\sqrt{mQ_\tau}}\frac{R}{2^{m-2}\,B\rho}\times$$

$$\left|\frac{1}{2\pi}\int_0^{2\pi}\frac{\beta_x^{m/2}}{(m-1)!}\frac{\partial^{m-1}B_z}{\partial x^{m-1}}\,e^{in\theta}\,d\theta\right|$$

The important difference from the smooth case is that the Fourier component of B_z is weighted by the β-function to the power $m/2$.

Integer resonance $m = 1$

$$\Delta A = \pi \frac{b_{n,1}}{\sqrt{Q_\tau}} = \frac{\pi}{\sqrt{Q_\tau}} \frac{\overline{R}}{\overline{B}} \frac{B_n}{Q}$$

This appears in early cyclotron theory because the $Q = 1$ resonance was used to extract the beam. See for example [Al Garren et al, Nucl. Instr. Meth. **18,19** (1962) p. 543].

Keil-Sessler doublet lattice: Worst (smallest) Q_τ appears to be ~ 1, at energy 20 GeV; $\epsilon_n = 1.7$ mm so $\epsilon = 10\,\mu$m, $A = 6$ mm and we would want ΔA less than about a tenth of this. $\overline{B} = B\rho/R = 1$ T. So we find

$$B_n < 400 \text{ Gauss},$$

where $n = Q$. This seems OK.

Half-integer, $m = 2$

The formula

$$\log \frac{A_{\mathrm{f}}}{A_{\mathrm{i}}} = \frac{\pi}{\sqrt{2}} \frac{b_{n,2}}{\sqrt{Q_\tau}}$$

was verified experimentally in the TRIUMF cyclotron [R. Baartman, G.H. Mackenzie, M.M. Gordon, 10^{th} Int. Conf. on Cyclotrons and Apps. (1984) p. 40]Link.

$b_{n,2} = \frac{\overline{R}}{n\overline{B}} \frac{\partial B_n}{\partial x}$, where $n = 2Q$, applied to the Keil-Sessler doublet FFAG and as before allowing only a 10% growth in amplitude, gives

$$\frac{\partial B_n}{\partial x} < 200\,\text{G/m}$$

Or, more precisely, for non-smooth case,

$$\left| \beta_x \frac{\partial B}{\partial x} \right|_{n=2Q} < 800\,\text{G}$$

1/3-integer, $m = 3$

$$\Delta\left(\frac{1}{A}\right) = \frac{\sqrt{3}\,\pi}{4}\,\frac{b_{n,3}}{\sqrt{Q_\tau}}$$

or, more precisely,

$$\Delta\,\epsilon^{-1/2} = \frac{\pi}{4\sqrt{3}}\,\frac{1}{\sqrt{Q_\tau}}\,\frac{R}{B\rho}\,\left|\beta_x^{3/2}\,\frac{\partial^2 B_z}{\partial x^2}\right|_{n=3Q}$$

The trend for resonances is that they are less dangerous the higher the order, so one would expect the imperfection third order resonances to be easily crossed. However, most proposed muon FFAGs traverse a cell tune of 1/3. This is the **intrinsic** resonance $3Q = N_{\text{cells}}$. Even a slight systematic sextupole component will result in very large $b_{n,3}$ when $n = N_{\text{cells}}$.