Integer Resonances and Closed Orbit Distortions (in synchrotons)

Maybe related to resonances crossing in FFAG synchrotrons

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FFAG workshop at TRIUMF

- Based on my elementary lectures at IHEP. Beijng, and JAERI, TOKAI
- Y. MORI's question
  Integer resonance $- X_{co} \to \infty$
  difficult to understand intuitively
  - initial-value problem ?
- ~~Synchro~~ Synchro betatron coupling by Dispersion $\to$ Satellite of integer $\to$ large D

◎ Point one

- mechanism of betatron resonances
  = same for both linear and
  nonlinear cases

◎ Point two

- analytical theory for integer resonance
  ΔB (field error)

  ① excites rigid dipole oscillations

  ② create C.O.D

  These two give the same result.

Naturally this is not new — H. G. HEREWARD

H.G. Hereward DESY PE-I
S-β resonance - 85/12 (1985)

When the particle goes through the cavities its energy is changed, and this shifts its equilibrium orbit. The betatron oscillations (about the equilibrium orbit) are excited in the same way that one can excite oscillations of a pendulum by moving its point of support. Simple calculation of the frequencies that come into play shows that there are satellite resonances at $\pm Q_s$ (and $\pm 2Q_s$, $3Q_s$...if the synchrotron oscillations are nonlinear) from the horizontal dipole, Q_H = integer, resonances.

The jump of equilibrium orbit at the cavities can be suppressed by putting them at a place with zero dispersion[1]. Then that term discussed above for driving the Q_H = integer $\pm$ n Q_s satellite resonances becomes zero. The question is whether other terms remain to drive them.

The conceptually simplest way to drive those satellites is to put in a momentum-dependant kick; the synchrotron oscillations modulate the momentum and that modulates the kick at Q_s (and maybe harmonics), ~~and~~ there appear the integer satellites. This momentum-dependant kick could for example be a bending magnet, with deflection proportional to $(1 + \frac{\Delta p}{p_o})^{-1}$. I am not concerned whether it is at a place in the lattice where there is dispersion, or not.

dipole kick

$D \frac{\Delta p}{p}$ に入る!

just a comment

At first sight it seems absurd to suggest that putting a bending magnet into a machine can make a whole family of satellites appear, because the ideal machine consists mainly of bending magnets already. So we have a paradox to resolve.This can be done by noting that there are two ways of dealing with an added dipole in a machine: (i) you can hold to the old reference orbit, then the dipole is a perturbation on every turn; or (ii) you can calculate the new closed orbit with the dipole in, and thereafter use it as reference orbit. The equations of motion relative to that reference orbit do not contain the dipole perturbation. The two methods are both perfectly valid but it is dangerous to mix them.

my view に近い

これが大事

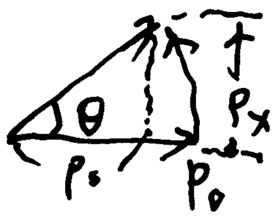
Consider this added bending magnet as a satellite generator by method (i). Suppose it makes deflection (radially outward) of

$$\Theta_o \left(1 + \frac{\Delta p}{p_o}\right)^{-1}$$

$\boxed{\text{Resonances}}$ = Point One

Where does the transverse energy come? (total energy const)

→ longitudinal energy



Kick!

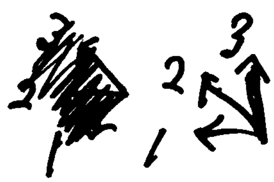
$$x' = \frac{p_x}{p_s} = \tan\theta$$

but

$$x' = \frac{p_x}{p_0} = \sin\theta$$

from Hamiltonian

p_0 : total momentam



Continuas Kick

→ resonance

mechanism is the same $= \begin{cases} \text{linear} \\ \text{nonlinear} \end{cases}$

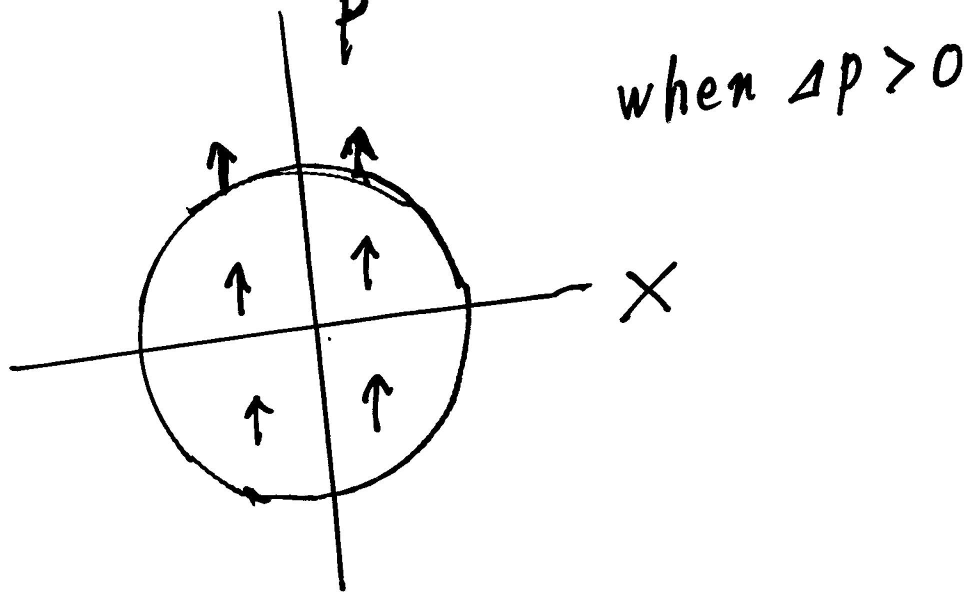
$\Delta x = 0$ (thin lens) $\Delta X = 0$

$\Delta x' \neq 0$

$\Delta p = \sqrt{\beta}\ \Delta x'$

Single kick - no x dependence

(Phase space)



rigid dispole motion

○ Point two

- How does the particles know D and x_{co}?

① Matrix

$$\begin{pmatrix} D \\ D' \\ \frac{\Delta P}{P} \end{pmatrix} = \underbrace{\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix}}_{\text{one turn}} \begin{pmatrix} D \\ D' \\ \frac{\Delta P}{P} \end{pmatrix}_0$$

$$\begin{pmatrix} x_{co} \\ x'_{co} \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{co} \\ x'_{co} \\ 1 \end{pmatrix}$$

similar

$\Updownarrow$

$$P_0 = eB_0\rho_0$$

$$\downarrow$$

$$x'' + K(s)\, x = \frac{1}{\rho(s)} \frac{\Delta P}{P} - \frac{1}{\rho_0} \frac{\Delta B}{B}$$

$$\begin{pmatrix} x_{co} \\ x'_{co} \end{pmatrix}, \begin{pmatrix} D \\ D' \end{pmatrix} = \frac{1}{2(1-\cos\mu)} \begin{pmatrix} e + bf - de \\ f + ce - af \end{pmatrix}$$

$$\Downarrow$$

$$e = (1-a)D - bD'$$

$$f = -cD + (1-d)D'$$

$$a = \cos\mu$$

$$b = \sin\mu$$

$$c = -\sin\mu$$

$$d = \cos\mu$$

$$x = a x_0 + b x'_0 + \{(1 - aD - bD'\} \frac{\Delta P}{P}$$

$$= \cos\mu\,\underbrace{\left(x_0 - D\frac{\Delta P}{P}\right)}_{x_{\beta 0}} + \sin\mu\,\underbrace{\left(x'_0 - D'\frac{\Delta P}{P}\right)}_{x'_{\beta 0}} + D\frac{\Delta P}{P}$$

$$\cancel{\Rightarrow D\frac{\Delta P}{P}(1-\cos\mu}$$

$\boxed{\begin{array}{l}\bullet \text{ Green function} - \text{variation of constants} \\ \qquad\qquad \text{initial-value approach}\end{array}}$

analytically tracking Brute force

- equation $\quad ' = \frac{d}{d\phi}$ Normalized Variables

$$X'' + r^2 X = f = \underbrace{\Delta\theta\, r\sqrt{\beta_0}}_{f} \quad \delta_p(\phi)$$

$$\Delta\theta = -\frac{\Delta B l}{B \rho}$$

- solution I

$$I = \frac{f}{r}\left[\int_0^{\phi} \sin r(\phi - \phi')\,\delta_p(\phi')\,d\phi'\right]$$

$$= \frac{f}{r}\Big[\sin r\phi \underbrace{\int_0^{\phi} \cos r\phi'\,\delta_p(\phi')\,d\phi'}_{B}$$

$$- \cos r\phi \underbrace{\int_0^{\phi} \sin r\phi'\,\delta_p(\phi')\,d\phi'}_{-A}\Big]$$

$$A = -\int_0^{\phi} \sin \nu \phi' \, \delta_p(\phi') d\phi'$$

$$\boxed{= -\sum_{n=0}^{N} \sin 2\pi \underset{\substack{\downarrow \\ \Delta\nu}}{\nu} n}$$

← Answer to Prof. Y. MORI

$$= \frac{\cos \pi\nu - \cos \pi\nu(2N+1)}{2\sin \pi\nu}$$

$$B = \int_0^{\phi} \cos \nu(\phi - \phi') \, \delta(\phi') d\phi'$$

$$\boxed{= \sum_{n=0}^{N} \cos 2\pi \underset{\substack{\downarrow \\ \Delta\nu}}{\nu} n}$$

$$= \frac{\sin \pi\nu + \sin \pi\nu(2N+1)}{2\sin \pi\nu}$$

single ΔB, N-turn

sum $\sum_{n=0}^{N} e^{2\pi i \nu n}$ (easy!)

$$X = A\cos\gamma\phi + B\sin\gamma\phi$$

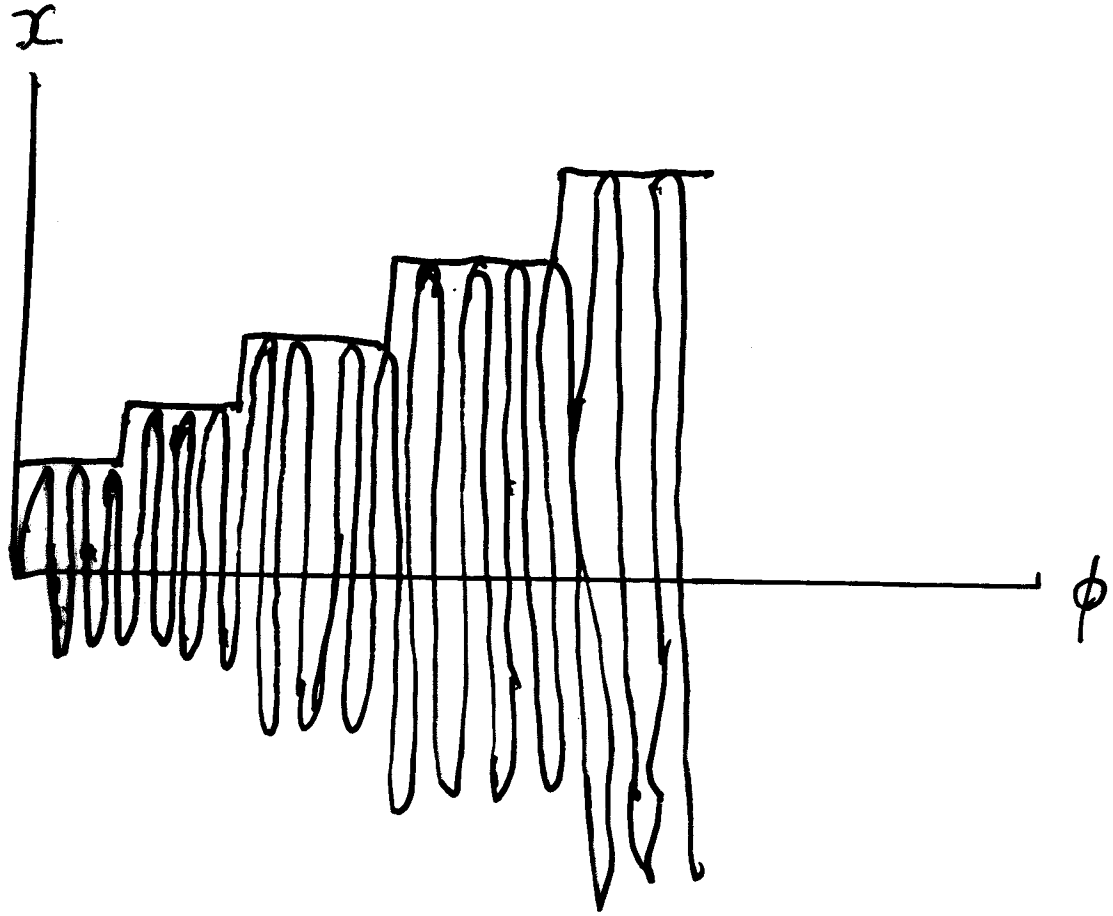
$$= \sqrt{A^2+B^2}\ \cos(\gamma\phi + \psi)$$

rapidly oscillating

slow function of ϕ

$\downarrow$

old resonant perturbation theory



$$X = A\cos\nu\phi + B\sin\nu\phi$$

$$X' = -\nu A\sin\nu\phi + \nu B\cos\nu$$

$$A'\cos\nu\phi + B'\sin\nu\phi = 0$$

same as the condition of variation of constants method

Observe at fixed point on the

$$\phi = 2\pi N$$

$$X = \frac{\Delta\theta\sqrt{\beta}}{2}\left[\cot\pi\nu - \cot\pi\nu\cos 2\pi\nu N + \sin 2\pi\nu N\right]$$

$$P = \frac{X'}{\nu} = \frac{\Delta\theta\sqrt{\beta}}{2}\left[-1 + \cos 2\pi\nu N + \cot\pi\nu\sin 2\pi\nu N\right]$$

$$X_{co} = \frac{\Delta\theta\sqrt{\beta}}{2}\cot\pi\nu$$

$$P_{co} = -\frac{\Delta\theta\sqrt{\beta}}{2}$$

$$\downarrow$$

$$x_{co} = \frac{\Delta\theta\,\beta}{2}\cot\pi\nu$$

$$x'_{co} = \frac{-\Delta\theta}{\sqrt{\beta}}\left(1+\alpha\cot\pi\nu\right) \qquad ' = \frac{d}{ds}$$

infront of Bend

after $x'_{co} \to x'_{co} + \Delta\theta$

$$x'_{co} = \frac{\Delta\theta}{2}\left(1-\alpha\cot\pi\nu\right)$$

Particular Solution

$$X = X_{co} - X_{co} \cos 2\pi\nu N - P_{co} \sin \pi\nu N$$

$$P = P_{co} - P_{co} \cos 2\pi\nu N + X_{co} \sin 2\pi\nu N$$

zero for $N=0$ / initial value

general solution of homogeneous equation

$$X_h = X_0 \cos 2\pi\nu N + P_0 \sin 2\pi\nu N$$

$$P_h = -X_0 \sin 2\pi\nu N + P_0 \cos 2\pi\nu N$$

$$X = X_h + X_{co}$$

$$= \underbrace{(X_0 - X_{co})}_{= X_{\beta 0}} \cos 2\pi\nu N + \underbrace{(P_0 - P_{co})}_{= P_{\beta 0}} \sin 2\pi\nu N + X_{co}$$

$$P = P_{co} + \underbrace{(P_0 - P_{co})}_{= P_{\beta 0}} \cos 2\pi\nu N - \underbrace{(X_0 - X_{co})}_{= X_{\beta 0}} \sin 2\pi\nu N$$