Building Taylor Maps with Mathematica and Applications to FFAG

Mathematica **can handle Taylor series analytically.**

All coeff. are numbers ⇒ numerical map.

For now only 4D map:

$$\boldsymbol{X_f = :\mathcal{M}: X|_{X=X_0} = e^{:F_{conc}:} R_{tot}.X|_{X=X_0}}$$

$\boldsymbol{X_0}$**=initial vector;** $\boldsymbol{X_f}$**=final vector.**

Phase sp. vector: X=(x, p_x, τ, p_τ) – **deviations from ref. orb. with momentum** p_0 **and curvature** $h(s) = 1/\rho_0(s)$**.**

Note: $:\mathcal{M}:$ **is just 4 polynomial functions of the comp. of X (and in fact 3, because** $p_t = const$)

<u>**Closed orb**</u> **as fixed point (**`FindRoot`**)**
<u>**tune**</u> **by Jacobian at the fix.pt.**
<u>$c\tau$</u> **directly from the 3-d polynom.**

Fixed point is a solution $(x, p_x)|_{co}$ **of the first two eqn.** $X_f = X_0$ **for a fixed** p_t**.**

FFAG opt. elements: combined sect. bend B
quadrupole Q
drift D

1 cell = **(Q-D-B-D-Q)**

Field expansion: We assume field B_y changes in radial dir. linearly to sec. order in x and keep only cubic terms in A_s and H)

$$\frac{eA_s}{p_0} = -hx + \frac{1}{2}(k_1 + h^2)x^2 - \frac{1}{6}(hk_1 - 3h^3)x^3 + O(x^4)$$

$$\frac{e}{p_0}B_y(x) = \frac{e}{p_0}\left(\frac{hA_s}{1+hx} + \frac{\partial A_s}{\partial x}\right) = -h + k_1 x + O(x^3)$$

The long. vect pot. is truncated to 3d order of x.

COSY uses a higher order expansion obeying LAPLACE – expect differences!

Hamiltonian before expansion of the $\sqrt{\ldots}$:

$$H = -(1+hx)\left[\frac{eA_s}{p_0} \quad + \quad \sqrt{(1+\delta)^2 - p_x^2}\right] =$$

geom.terms kinematic

$$= hx + \frac{1}{2}(h^2 - k_1)x^2 - \frac{1}{3}hk_1x^3 + 0(x^4) -$$

$$-(1+hx)\sqrt{(1+\delta)^2 - p_x^2} - \frac{p_\tau}{\beta_0}, \quad where$$

$$(1+\delta)^2 = 1 - \frac{2p_\tau}{\beta_0} + p_\tau^2; \qquad \delta = (p - p_0)/p_0.$$

All momenta normalized to p_0.

now expand H over X_i to order Nord (`Series`):

for geom. terms retain order 3

for kinematic retain $Nord >> 1$

Detour: single combined sector bend

The rotational symmetry implies that all closed orbits are circular. This follows from one of the equations of motion taken with the following constraint – the off-momentum (p) orbit has to be at each point parallel to the reference (p_0) orbit:

$$p'_x = -\frac{\partial H}{\partial x}|_{\,p_x=0} = 0 \qquad \Rightarrow$$

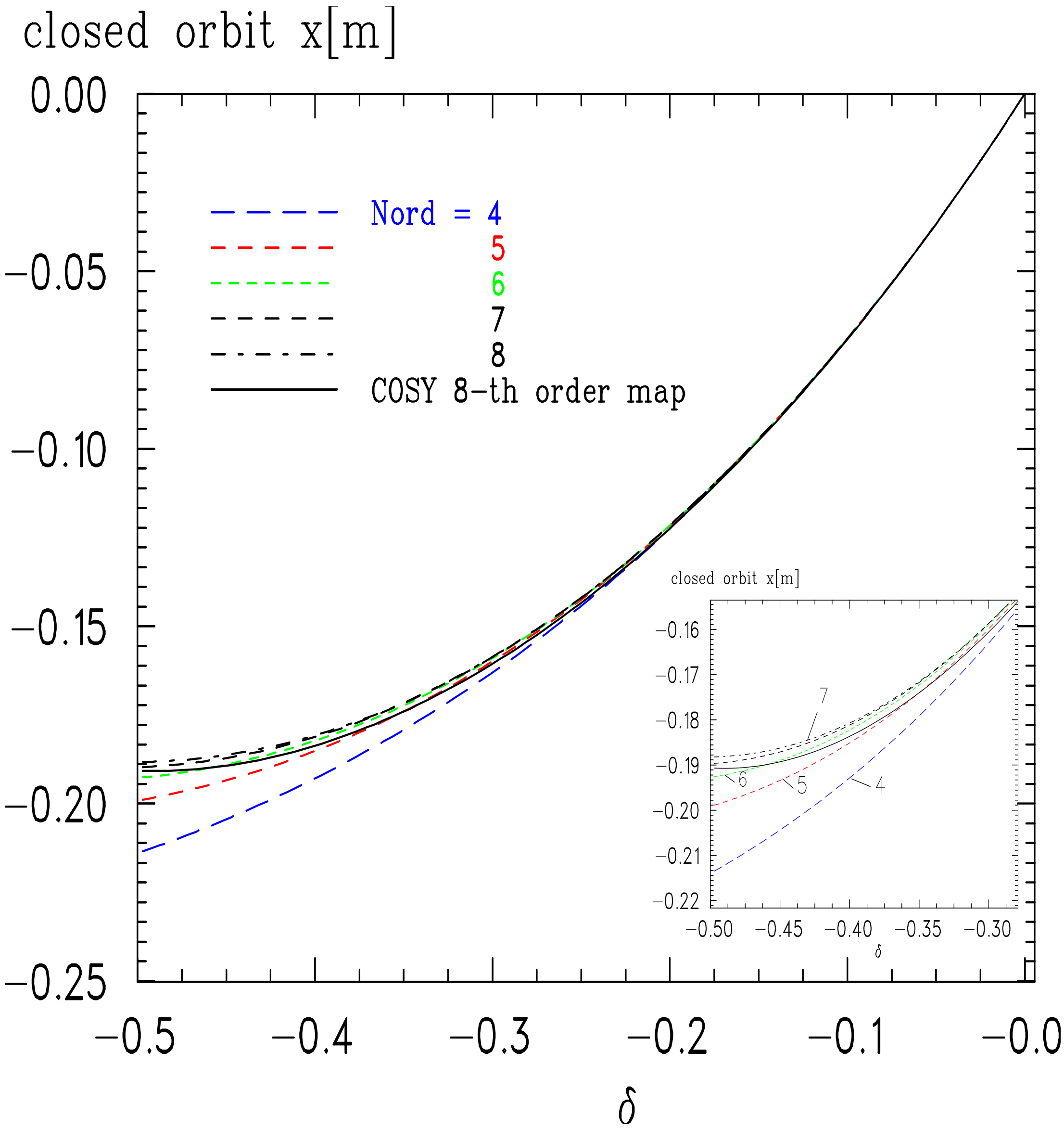
$$h^2 x - k_1 x - h k_1 x^2 - h\delta = 0 \qquad \Leftrightarrow$$

$$e(B_0 + \frac{n_0 B_0}{\rho_0} x)(\rho_0 + x) = p_0(1+\delta) = p\ .$$

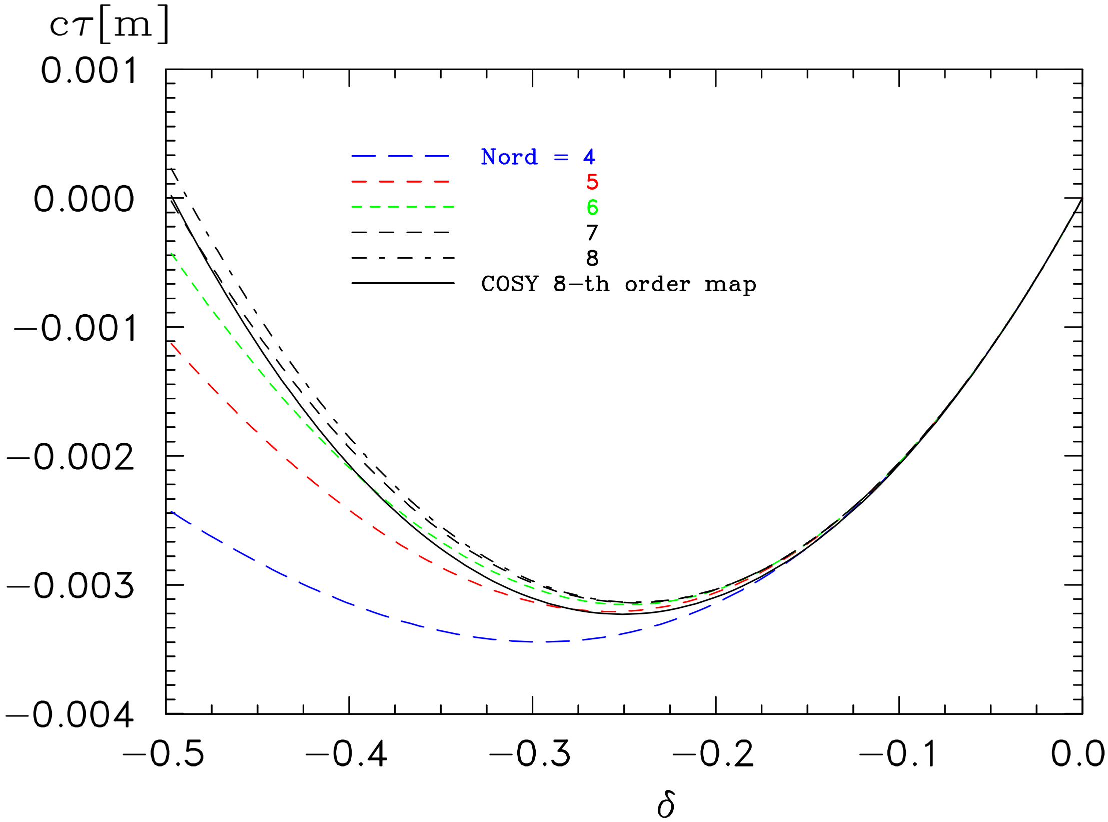
($'$) denotes derivative w.r.t. s. The off-momentum closed orbit is arc of a circle of radius $\rho_0 + x$, at which radius the magnetic field is $B_0 + n_0 B_0/\rho_0 x$.

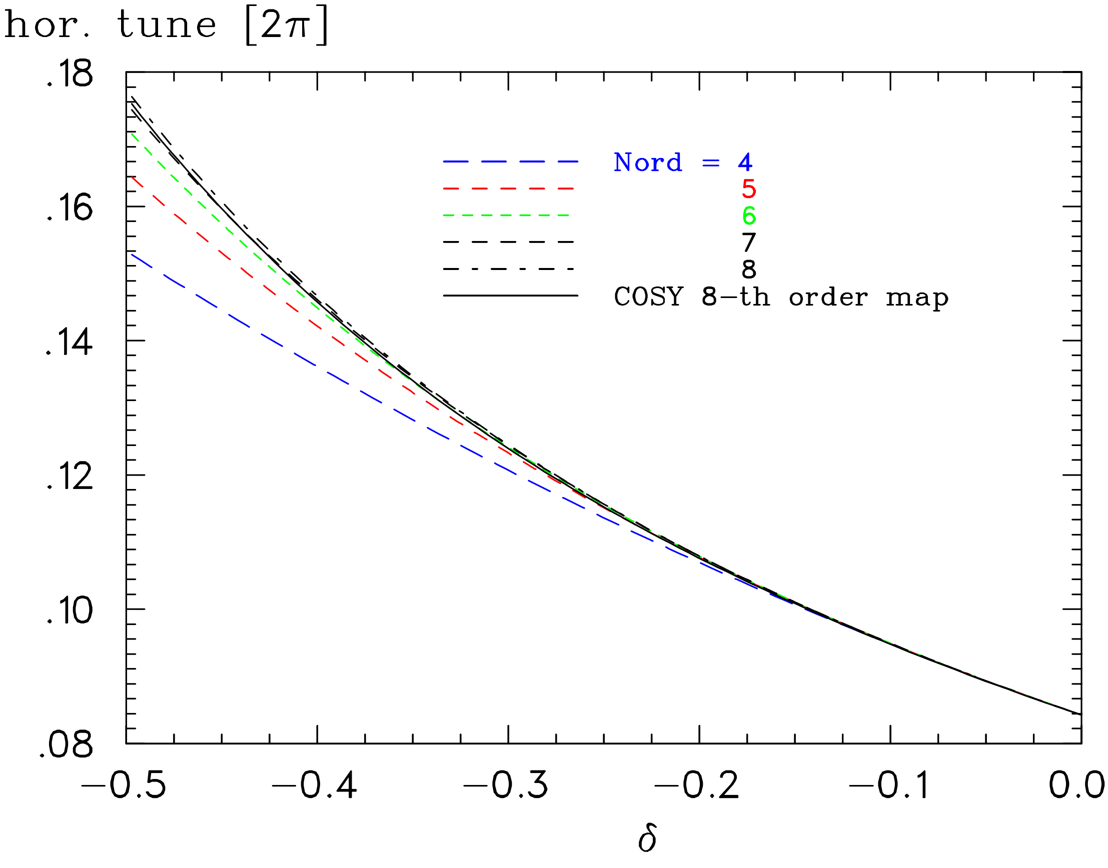
Here $h(s) = 1/\rho_0(s)$ and $p_0 = eB_0\rho_0$ are the design orbit curvature and momentum, and $-k_1\rho_0^2 \equiv n_0$ is the field index.

Comparisons: Lie algebra notebook and COSY



COSY script – orbit finder is a courtesy of
Dejan Trbojevic


cτ[m]
0.001
0.000
−0.001
−0.002
−0.003
−0.004
Nord = 4
5
6
7
8
COSY 8−th order map
−0.5
−0.4
−0.3
−0.2
−0.1
0.0
δ


hor. tune [2π]
.18
.16
.14
.12
.10
.08
Nord = 4
5
6
7
8
COSY 8−th order map
−0.5
−0.4
−0.3
−0.2
−0.1
0.0
δ

these operate on polynomials:
Poisson bracket (PB),
the Cambell-Baker-Hausdorf theorem (CBH),
the exponential series defining the Lie transform
and the module `GetRmat`.

$$\mathbf{H = PB[F, G]} \quad \Leftrightarrow h = \ : f : g \equiv [f, g] \, ;$$

$$\mathbf{H = CBH[F, G]} \ \Leftrightarrow h = f + g + \frac{1}{2} [f, g] + ...$$

$$(e^{:f:} e^{:g:} = e^{:h:})$$

$$\mathbf{H = LieExp[F, G]} \Leftrightarrow \ h = g + [f, g] + \frac{1}{2!} [f, [f, g]] + ...$$

$$(h = e^{:f:} g)$$

$$\boldsymbol{R} = \text{GetRmat}[f^{(2)}] \Leftrightarrow \ e^{:f^{(2)}:} \leftrightarrow R$$

F,G and H are polinomials (not monomials – the orders are not separated) R is 4x4 matrix.

A. Chao, Lecture Notes on Topics in Accel. Physics, Chapter 8: "Truncated Power Series Algebra" http://www.slac.stanford.edu/ achao/lecturenotes.html

```
PB[F_,G_] := (D[#1,x]D[#2,px]-D[#2,x]D[#1,px]+
+ <same for t,pt> &)[F, G];

CBH[F_,G_] := (F + G + 1/2 PB[F, G] + ...
+ 1/24 PB[F, PB[G, PB[G, F]]]) &[F, G];

LieExp[F_,G_] := (#2 + PB[#1, #2] +
1/2!*PB[#1, PB[#1, #2]] + ... ) &[F, G];
```

```
GetRmat[f2_] :=
 Module[{R, G, FF}, FF =
   ( -2 Coefficient[f2, x^2]  □  □  □ )
   (           □              □  □  □ )
   (           □              □  □  □ );
   (           □              □  □  □ )
  G = S.FF;
  R = MatrixExp[G] // ExpToTrig]
```

Nord = the order of the final map.
After each call of `CBH`, the resultant polynomial is truncated, retaining terms up to order Nord+1

Max. number of nested Poisson brackets determined empirically – to ensure accuracy to this order.

The same is valid for `LieExp`, but the retained terms are of order Nord.

Module `GetRmat` (Appendix) computes R – the transport matrix corresponding to linear operator $f^{(2)}$.

Other standard *Mathematica* operations are: `MatrixExp`, `Series`, `Chop`, `Coefficient`,`FindRoot` **and** `Timing`**.**

Algorithm to build the map

Reordering rule (earlier elements appear on the left – same arg X everywhere):

$$:\mathcal{M}: = \prod_{n=1}^{N_{ele}} e^{:f_n(X):}, \quad :f_n(X): \; = -L_n :H_n(X):$$

Thin kick factorization – present map as nonlin. kicks and linear operators (matrices)

$$e^{:f_n:} = e^{:f_n^{kick}:} e^{f_n^{(2)}},$$

which can be seen as a thin kick

$$e^{:f_n^{kick}:} = e^{:f_n:} e^{-:f_n^{(2)}:}$$

at the entrance of the n-th element.

Nele similarity transforms to commute all linear operators to the right, result in:

$$:\mathcal{M}: \; = \prod_{n=1}^{N_{ele}} e^{:f_n^{kick}(\tilde{R}_n . X):} R_{tot} = e^{:F_{conc}:} R_{tot}$$

where $\tilde{R}_n = \prod_k R_k$ is the accumulated matrix to the n-th kick and $R_{tot} = \tilde{R}_{Nele}$.

Finally:

$$X_f = \, :\mathcal{M}: X|_{X=X_0} = e^{:F_{conc}:} R_{tot} . X|_{X=X_0}$$

X_0=initial vector; X_f=final vector.

Results, CPU Time

- **We have compared off-energy cl. orb., hor. tune and orbit path length with the ones of COSY 8-th ord. map**

- **We expect that as N increases**
  N=4 $\rightarrow$ 8 $\Rightarrow$ better agreement
  this is seen to be true for N=4,5,6
  exact agreement is not possible
  (1-2 mm difference at $\delta = -0.5$)
  most likely caused by the missing geometric > 3 terms in A_s

- **Notes:**
  – **easy to extend to 6D, but time...;**
  – **cannot compete with COSY in speed, e.g.:**
  7-th order 4D map for 5 element line takes ~ 2000 sec on 1 GHz CPU

Table 1: approx result of `Timing` command (CPU time) in sec. (Numb. of nested P. brackets kept: 4 in LieExp and 3 in BCH)

Map ord.	concatenation loop	LieExp
4	25	40
5	120	150
6	300	450
7	2000	2000
8	2500	2550

References

[1] R.Ruth, AIP Conference Proc. 153, Vol.1 p. 166 Hamiltonian

[2] Johan Bengtsson, Doctorate Thesis CERN 88-05. Hamilt., $A_s(x, y)$ expansion

[3] A. Dragt, Exact Numerical Calculation of Chromaticity in Small Rings, Particle Accelerators, 1982, Vol 12 pp 205-218. geom. and kinem. terms both important in small rings

[4] A. Chao, Lecture Notes on Topics in Accel. Physics, Chapter 8: "Truncated Power Series Algebra" http://www.slac.stanford.edu/ achao/lecturenotes.ht the Lie algebra needed here

[5] A. Dragt and E. Forest, Computation of nonlinear behaviour of Hamiltonian systems using Lie algebraic methods J. Math. Phys. 24 (12), 1983. general

[6] A. Dragt, Lectures on Nonlinear Orbit Dynamics, AIP Proc. 87, Phys. High Energy Accel., Fermilab, 1981, p.147 Hamilt. terms for a bend (polar)

[7] J. Irwin, Construction of High-order Maps for Large Proton Accelerators; Tanaji Sen, Y.T. Yan, J. Irwin, LIEMAP : A Program for Extracting a one-turn Single Exponent Lie Generator Map,

Proc. PAC 1991 similar recipie for the map, but drift is linear

[8] **E.Forest, AIP Conf. Proc, 184 Vol 1 1987; also: E.Forest, R.D.Ruth, Fourth Order Symplectic Integration, Physica D vol. 43 (1990) 105-117 43 (Sect. 5.5: with the square root expanded to second order in p_x, p_y, and with cubic and higher terms in A_s ignored, the motion is exactly solvable giving rise to a δ-dependent R matrix).**

[9] **COSY Infinity version 7, K. Makino and M. Berz, in AIP Conference Proceedings, 391 (1996) 253. and** `http://bt.pa.msu.edu/pub/`

The author is thankful to Martin Berz for his comment on the final representation of the map