Circumference change for parabolic momentum dependence

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- The total circumference change over the acceleration range can be explicitly derived $$\Delta C = C_{\max} - C_{\min}$$
  - using the thin-lens equations and exploring the focal length dependency on momentum

  or
  - by exploiting the strong parabolic dependence of the circumference change and the symmetry imposed by the choice of lattice parameters:

$$for\ a\ symmetric\ parabola:$$

$$C_{\max} = C_u = C_l\ ; \quad C_{\min} = C_c$$

where C_{max} = circumference at $p^{u(pper)}$, $p^{l(ower)}$, and

C_{min} = circumference at $p^{c(enter)}$,

or the central energy of the cycle

- We know from the parabolic dependency of the circumference the explicit dependence of α (and therefore η) on δ must be

$$\alpha \cong \alpha_0 + \alpha_1 \delta$$

$$\eta \cong \eta_0 + \eta_1 \delta \qquad \text{for constant } \rho$$

- This implies

$$|\Delta C| = C_{\max}(-\alpha_0^l + \alpha_1^l \delta^l)\delta^l = C_{\max}(\alpha_0^u + \alpha_1^u \delta^u)\delta^u$$

- Where the coefficients correspond to momentum compaction at the lower and upper momentum, respectively and δ is taken as the momentum offset to the central energy, or bottom of the parabola. α coefficients are kept positive so that a negative momentum compaction is clearer

- or

$$\alpha_0^l = \alpha_0^u \frac{-\delta^u}{\delta^l} \; ; \; \alpha_1^l = \alpha_1^u \left(\frac{\delta^u}{\delta^l} \right)^2$$

- At transition (the bottom of the parabola)

$$\frac{\partial \Delta C}{\partial \delta} = C_{\max} (-\alpha_0^l + 2\alpha_1^l \delta^l) = 0$$

$$\alpha_0^l = 2\alpha_1^l \delta_c^l ; \; also \; \alpha_0^u = -2\alpha_1^u \delta_c^u$$

$$for \; \delta_c^l = \frac{p^c - p^l}{p^l} \; ; \; \delta_c^u = \frac{p^c - p^u}{p^u}$$

- For our rings with $p^u/p^l = 2$, $\delta^l = 0.5$, $\delta^u = -0.25$,

$$\alpha_0^l = \alpha_1^l \; and \; \alpha_0^u = 0.5\alpha_1^u$$

- This implies
  - One can study the first coefficient, α_0, to determine the behavior of ΔC as a function of free parameters
  - One can study the behavior at one momentum; the lower limit of stability, for example
    - which makes the problem substantially easier to parameterize

Circumference change in the thin-lens model

- In the thin-lens model, the total change in circumference (and therefore the total phase slip, or ΔT) can be estimated from the solutions found for dispersion about the central orbit ($\delta=0$). (The contributing term in δ is managed by symmetrizing the parabola.)

For a periodic ring

$$\Delta C = C\alpha_{ring}\delta = NL_{cell}\alpha_{cell}\delta$$

$$with \quad \alpha_{cell} = \frac{1}{L_{cell}}\int\frac{\eta}{\rho}ds$$

In the thin lens, η is linear across the half cell:

$$\eta_0 = \frac{(\eta_{\max} - \eta_{\min})}{D}s$$

D is the half cell length and $s \leq D$ is the distance from the center of the CF quadrupole

- Integrating over the length of the CF quadrupole, l_B, (for a half cell) and noting that in the thin-lens limit of stability $\eta_{min}=0$

$$\alpha_{0_{cell}} = \frac{1}{L_{cell}} \int_0^{l_B} \frac{\eta_0}{\rho} ds = \frac{1}{L_{cell}} \int_0^{l_B} \frac{\eta_{\max}}{\rho} \frac{s}{D} ds$$

$$= \frac{1}{L_{cell}} \frac{\eta_{\max} l_B^2}{2\rho D}$$

$$|\Delta C| = 2N\delta_c^l \left(\alpha_0^l - \tfrac{1}{2}\alpha_0^l\right) = N\delta_c^l \frac{\eta_{\max}}{2D} \frac{l_B^2}{\rho}$$

since $f^* = D$ at p^l for the lower limit of stability

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- Inserting $\eta_{max} = \theta_B f^*$ and $\theta_B = l_B / \rho$

$$|\Delta C| = N\delta_c^l \frac{\eta_{\max}}{2D} \frac{l_B^2}{\rho}$$

$$|\Delta C| = N\delta_c^l \frac{\vartheta_B^l f^*}{2D} \vartheta_B^l l_B = \frac{N\delta_c^l}{2} \vartheta_B^{l\,2} l_B$$

since $f^* = D$ at p^l for the lower limit of stability

$$\Delta C = \frac{N\delta_c^l}{2} {\vartheta_B^l}^2 l_B = \frac{N}{2}\left(\frac{p^c - p^l}{p^l}\right)\left(\frac{p^u}{p^l}\right)^2 \vartheta_B^2 l_B$$

since $\vartheta_B^l = \frac{p^u}{p^l}\vartheta_B^u = \frac{p^u}{p^l}\vartheta_B$

where ϑ_B is now the bend for the circular reference orbit

which is at p^u for a FDF and a symmetric parabola.

For the circular orbit :

$$\vartheta_B = \frac{\pi}{N}$$

$$\Delta C = \frac{\pi^2}{2}\left(\frac{p^c - p^l}{p^l}\right)\left(\frac{p^u}{p^l}\right)^2 \frac{l_B}{N} = \frac{\pi^2}{4}\left(\frac{p^u}{p^l} - 1\right)\left(\frac{p^u}{p^l}\right)^2 \frac{l_B}{N}$$

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- For cost, you desire N small, so cost and time of flight/circumference change are opposing conditions. The best you can do is try and minimize l_B and N simultaneously

$$|\Delta C| = \frac{\pi^2}{4}\left(\frac{p^u}{p^l}-1\right)\left(\frac{p^u}{p^l}\right)^2 \frac{l_B}{N}$$

- Further near the acceleration range we have chosen, 2:1, the dependence goes as the square of the range. The original designs were ≥3, with either the ΔC or the circumference a factor of 9 larger. This represents the biggest factor in the phase-slip profile/design.

- One can immediately draw certain conclusions:

$$\Delta C = \frac{\pi^2}{4}\left(\frac{p^u}{p^l}-1\right)\left(\frac{p^u}{p^l}\right)^2 \frac{l_B}{N} = \frac{\pi}{4}\left(\frac{p^u}{p^l}-1\right)\left(\frac{p^u}{p^l}\right)^2 \frac{p^u}{0.3}\frac{\vartheta_B^2}{B_B}$$

for a fixed acceleration range

$$\propto p^u \frac{\vartheta_B^2}{B_B} \propto p^u \frac{1}{N^2 B_B} = \left(\frac{p^u}{p^l}\right) p^l \frac{1}{N^2 B_B} \propto p^l \frac{1}{N^2 B_B}$$

this indicates for a fixed number of cells, the phase slip is minimized by the largest B_B in the CF magnet. It also shows a p^l dependence which suggests the source of the problem at the lower energies.

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- Naively one would hope that circumference would scale with momentum. However, we know that ΔT or ΔC must be held at a certain value for successful acceleration. If ΔC is set or scaled relative to the High Energy Ring (HER), then a Low Energy Ring (LER) would follow:

$$\Delta C \propto p^u{}_{(HER)} \frac{1}{N_H{}^2 B_B} = p^u{}_{(LER)} \frac{1}{N_L{}^2 B_B}$$

$$or \;\; N_L = \frac{N_H}{\sqrt{R}} \quad rather\ than \quad N_L = \frac{N_H}{R}$$

$$with \;\; R = \frac{p^u{}_{(HER)}}{p^u{}_{(LER)}}, \quad the\ scaling\ ratio$$

- Another approach is to fix the magnet aperture, Δx, and solve for θ_B. The two conditions are:

$$1) \qquad \Delta x = L_{1/2} \Delta \vartheta_B$$

$$2) \qquad \vartheta_B = \vartheta_B^l - \vartheta_F^l$$

- Where $\Delta\theta_B$ is the difference in bend angle in the CF magnet between the circular orbit at p^u and the low momentum orbit and $L_{1/2}$ is the length of the half cell. The second condition is geometric closure of the low-momentum reference orbit, the net bend is equal to the total bend/half cell of the circular orbit.

- Equation 2) can also be expressed as:

$$\vartheta_F^l = \vartheta_B^l - \vartheta_B = \left(\frac{p^u}{p^l} - 1 \right) \vartheta_B$$

- Inserting magnet lengths and inter-magnet spacing, L_M, gives

$$\Delta x = (l_B + l_F + l_M)\Delta\vartheta_B = (l_B + l_F + l_M)(\vartheta_B^l - \vartheta_B)$$
$$= (l_B + l_F + l_M)\left(\frac{p^u}{p^l}\vartheta_B - \vartheta_B\right) = (l_B + l_F)\left(\frac{p^u}{p^l} - 1\right)\vartheta_B$$

for L_M small compared to the half cell length

- Converting lengths to fields and angles gives

$$\Delta x = \left(\frac{p^u}{p^l} - 1\right)\vartheta_B\left[\frac{p^u\vartheta_B}{0.3B_B} + \frac{p^l\vartheta_F^l}{0.3B_F}\right]$$
$$= \left(\frac{p^u}{p^l} - 1\right)\vartheta_B\left[\frac{p^u\vartheta_B}{0.3B_B} + \frac{\left(p^u - p^l\right)\vartheta_B}{0.3B_F}\right]$$

- We are looking for approximate scaling laws, and $B_F \sim B_B$,

$$\Delta x \approx \left(\frac{p^u}{p^l}-1\right)\left[\frac{\left(2p^u - p^l\right)}{0.3B_B}\right]\vartheta_B^2$$

$$\vartheta_B \approx \frac{\sqrt{0.3B_B \Delta x}}{\sqrt{p^l}} \frac{1}{\sqrt{(c-1)(2c-1)}} \quad for \quad \frac{p^u}{p^l}=c$$

- The square root dependence on momentum reappears indicating again

$$\frac{N_L}{N_H}=\frac{\vartheta_H}{\vartheta_L}=\sqrt{\frac{p^{l}{}_{(L)}}{p^{l}{}_{(H)}}}=\frac{1}{\sqrt{R}}$$

- As pointed out by Scott, Shane … what you really want is ΔC/N to remain constant
- The scaling law is then approximately:

$$N_L \approx \frac{N_H}{\sqrt[3]{R}}$$

- This is somewhat optimistic because you are simply keeping the number of turns, and ΔT ~ constant.
- For our rings this implies the 2.5-5 GeV ring is only ~60% the size of th 10-20 GeV ring. Scott's optimizer finds 80% so this is fairly close for an approximate description

Choice of technical parameters=lattice

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- If one chooses
  - Δx between outer and inner orbits; i.e. magnet aperture
  - drifts
  - Phase advance
  - Momentum range
- The optics are completely determined by
  - The peak fields in the two quadrupoles
  - Geometric closure of off-momentum and on-momentum orbits
- It follows that θ_B can be expanded completely in terms of fields in the focusing and CF magnets. If the circumference change or phase slip is too large, then the magnet aperture is the parameter which should be reduced; this reduces θ_B and increases the # of cells.

Choices in magnet parameters

- There are 3 regimes of magnet design:

7T, 4.5-2T, 1.5T

- There is a discrete jump at ~4.5 T in the cost of a superconducting magnet as the design goes to a single-turn coil. For the same aperture, 2-4T magnets are roughly equivalent. *verify

- A large-aperture 7T magnet is probably prohibitively expensive
- It is likely that the optimum for the two higher energy rings may be 4T magnets
- The circumference of the lower energy ring appears to be cost prohibitive. A comparable-size ring with cheaper magnets? The next jump in magnet cost would be large-aperture normal conducting and pulsed, 1.5T. (Refer to the large-aperture Fermi proton driver design.)