Performance Criteria and Optimization of FFAG lattices

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Minimum Momentum-compaction lattices for nonscaling FFAGs

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- Minimizing the overall phase slip between the rf waveform and the beam due to pathlength differences as a function of momentum, is important when rapid acceleration precludes adjustments to the rf phase. Phase slippage of reference orbits can be described as a change in circumference:

$$\frac{\Delta C}{C} = \alpha_{ring} \frac{\Delta p}{p} = \alpha_{ring}\delta$$

$$\alpha_{ring} = \frac{1}{C}\int \frac{\eta}{\rho} ds$$

- It is clear that minimizing the integrated dispersion function in the areas of dipole bend fields is the important optimization criterion for controlling phase slip. This optimization refers to a particular circular reference orbit.

- Since the FFAG lattices are completely periodic, the total ΔC is $N \times L_{cell}$ $(\alpha_{cell}\,\delta)$ where N is the number of cells. Since $N \times L_{cell} = C$, $\alpha_{ring} = \alpha_{cell}$.

Momentum Compaction in a non-scaling FODO FFAG

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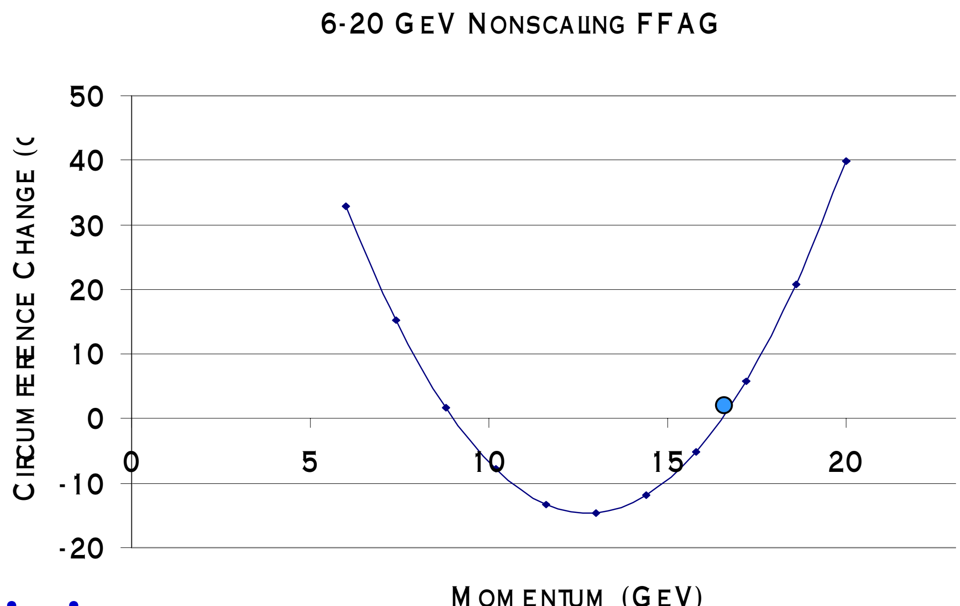
- As is now well known, the change in revolution time or change in circumference is parabolic. If the lattice is properly optimized, it is symmetric about the median energy in the acceleration cycle. (A non-optimized lattice produces an asymmetric parabola, and increased |ΔC|, top to bottom.)
- The momentum compaction is well described

by:

$$\alpha \cong \alpha_0 + \alpha_1 \delta$$

$$\therefore\ \eta \cong \eta_0 + \eta_1 \delta$$

$$\text{for constant } \rho$$



Optimization approach:

Symmetric parabola + Minimize α_o through η_o

Momentum Compaction in a non-scaling FODO FFAG

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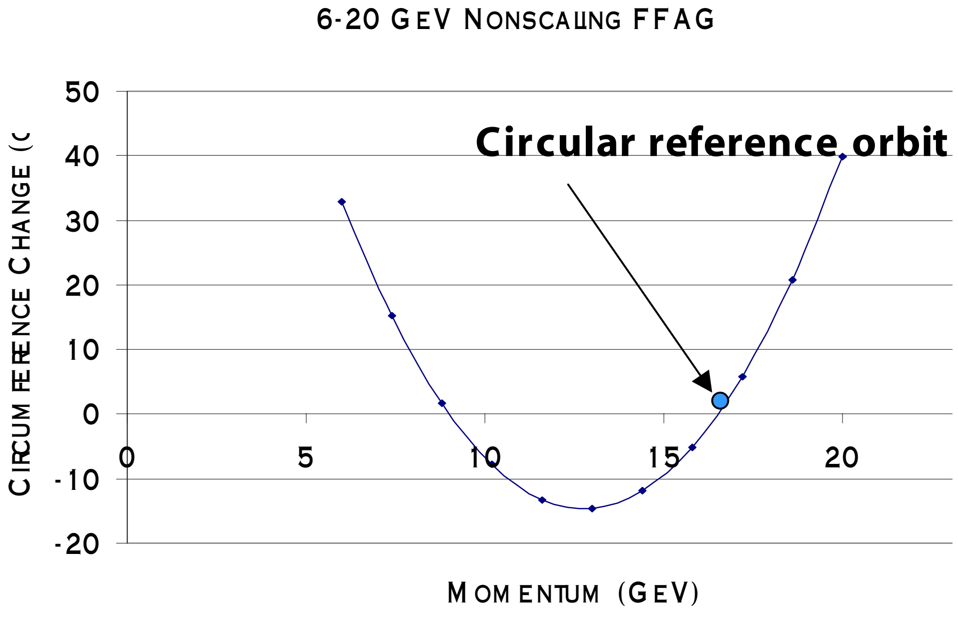
- Since optical parameters and orbits change with energy, optimization occurs relative to an appropriately chosen orbit, defined as the one which traverses the center or 0-gradient point of both quadrupoles
- This "central" or circular orbit is chosen to symmetrize the parabola and minimize the total swing of $|\Delta C|$.



- This is the design tool used to control the impact of the δ terms in α
- An optical structure analysis will be used to optimize/minimize α_0

Linear Dispersion in thin-lens FODO optics

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- Since α is dependent on dispersion, dispersion will first be studied and minimized in FODO optics.
- Dispersion can be expressed in standard thin-lens matrix formalism.

$$\begin{bmatrix} x \\ x' \\ \delta \end{bmatrix} = \begin{bmatrix} \eta\delta \\ \eta'\delta \\ \delta \end{bmatrix} = \begin{bmatrix} \eta \\ \eta' \\ 1 \end{bmatrix} = \mathbf{M} \begin{bmatrix} \eta_0 \\ \eta_0' \\ 1 \end{bmatrix}$$

- At the symmetry points of the FODO cell the slope of optical parameters is zero, and correspond to points of maximum and minimum dispersion. For horizontal dispersion, the center of the vertically-focusing element is a minimum and horizontally-focusing element is a maximum.

$$\begin{bmatrix} \eta^{\min} \\ 0 \\ 1 \end{bmatrix} = \mathbf{M}^{1/2FODO} \begin{bmatrix} \eta^{\max} \\ 0 \\ 1 \end{bmatrix}$$

Thin lens matrix solutions for different field options in a FODO

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- The transfer matrix for a bend field centered in the drift between focusing elements: 1/2F-drift-1/2D is:

$$\mathbf{M}^{1/2FODO} = \begin{bmatrix} 1 & 0 & 0 \\ 1/f & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & L/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \vartheta_B \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & L/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/f & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - L/f & L & \frac{1}{2} L \vartheta_B \\ -L/f^2 & 1 + L/f & \vartheta_B (1 + \frac{1}{2} L/f) \\ 0 & 0 & 1 \end{bmatrix}$$

- For the bend field centered in the vertically-focusing element:

$$\mathbf{M}_{CF}^{1/2FODO} = \begin{bmatrix} 1 - L/f & L & 0 \\ -L/f^2 & 1 + L/f & \vartheta_B \\ 0 & 0 & 1 \end{bmatrix}$$

Dispersion and dipole location

- the dispersion solution for the first case

$$\eta^{\max} = \frac{f^2}{L}\vartheta_B\left(1+\frac{1}{2}\frac{L}{f}\right)$$

$$\eta^{\min} = \frac{f^2}{L}\vartheta_B\left(1-\frac{1}{2}\frac{L}{f}\right)$$

- The dispersion solution for the bend field located in the vertically-focusing element is clearly less indicating smaller pathlength variations:

$$\eta_{CF}^{\max} = \frac{f^2}{L}\vartheta_B$$

$$\eta_{CF}^{\min} = \frac{f^2}{L}\vartheta_B\left(1-\frac{L}{f}\right)$$

Minimizing momentum compaction in a FODO cell

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- Minimizing the dispersion function requires centering the dipole field about low β_x, low β_x is also the position of η^{min} in a periodic structure (center of vertically-focusing quad!) this positioning, minimizes the momentum compaction:

$$\alpha = \langle \eta^{min} \rangle / \rho \qquad \textit{(thin lens)}$$

$$\alpha \propto \langle \eta^{max} + \eta^{min} \rangle \qquad \textit{(current lattices)}$$

- It is generally true that in periodic optics:

minimizing $\eta \equiv$ minimizing α

- In addition, $\vartheta_B = N/4\pi$ (half-cell), so α is dependent on the focal length and the number of cells; the circumference change which determines the phase slip is therefore,

$$\Delta C = N \times \alpha_{cell} \delta L_{cell} \propto \frac{f^2}{L} N^2 \approx f\, N^2$$

- The focal length scales with half cell length so the dependence is really linear. The focal length dependence is important in discriminating between optical structures and optimizing the lattice.

Split FODO, or triplet FFAG

- An structure defined as FDF [1/2rfdrift-QF—short drift—QD-short drift-QF-1/2rf drift] has been demonstrated to provide a significantly reduced phase slip relative to an equivalent FODO cell, where equivalent is defined in terms of rf drift length, total bend angle per cell, intermagnet spacing and pole-tip fields.
- Understanding the improvement leads back to the original thin lens matrices. For an rf drift inserted at the center of the horizontally-focusing quadrupole:

$$\mathbf{M}_{CF}^{1/2FDF} = \begin{bmatrix} 1-D/f_1 & D & 0 \\ -1/f^* & 1+D/f_2 & \vartheta_B \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & L_{rf}/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Where D, the distance between quadrupole centers, replaces the half-cell length

$$= \begin{bmatrix} 1-D/f_1 & \frac{L_{rf}}{2}\left(1-D/f_1\right)+D & 0 \\ -\frac{1}{f^*} & -\frac{L_{rf}}{2}\left(\frac{1}{f^*}\right)+\left(1+D/f_2\right) & \vartheta_B \\ 0 & 0 & 1 \end{bmatrix}$$

Transfer matrices for split FODO cells

- Adding the drift at the center of the vertically focusing quadrupole instead gives:

$$\mathbf{M}_{CF}^{1/2DFD} = \begin{bmatrix} 1 & L_{rf}/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1-D/f_1 & D & 0 \\ -\frac{1}{f^*} & 1+D/f_2 & \vartheta_B \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \left(1-D/f_1\right)+L_{rf}/2\left(-\frac{1}{f^*}\right) & D+L_{rf}/2\left(1+D/f_2\right) & \vartheta_B\, L_{rf}/2 \\ -\frac{1}{f^*} & \left(1+D/f_2\right) & \vartheta_B \\ 0 & 0 & 1 \end{bmatrix}$$

- Note that the half cell contains only half the rf drift, hence the added drift matrix is $L_{rf}/2$, rather than the half-cell length as in the FODO cell case.

Dispersion function for split FODO

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- The combined focal length, is the general result for a doublet quadrupole lens system.

$$\eta_{FDF}^{max} = \vartheta_B f^*$$

$$\eta_{DFD}^{max} = \vartheta_B f^*$$

$$\eta_{FDF}^{min} = \left(1 - D/f_1\right)\eta_{CF}^{max} = \vartheta_B f^* \left(1 - D/f_1\right)$$

$$\eta_{DFD}^{min} = \left(1 - D/f_1 - \frac{L_{rf}}{2}\frac{1}{f^*}\right)\eta_{CF}^{max} + \vartheta_B L_{rf}/2$$

$$= \vartheta_B f^* \left(1 - D/f_1\right)$$

$$where \quad \frac{1}{f^*} = \frac{1}{f_1} - \frac{1}{f_2} + \frac{D}{f_1 f_2}$$

- With the rf drift placed at the center of the horizontally focusing element (leftmost equations), or the vertically one (rightmost equations). The equations are the same so the differences between them and from the FODO cell are not immediately obvious we unless we explore the possible values for f_1 and f_2.

Split-cell focal lengths

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- One can solve for these focal lengths in the limits of stability and use their relative scaling over the entire acceleration range as a basis for comparison with other lattice models also explored in their limit of stability.
- In the presence of no bend, 90 degrees of phase advance across a half cell represents the limit of stability for FODO-like optics (single minimum). This implies for a initial position on the x axis (x,x'=0), that its position will be 0 (x=0,x') after a half-cell transformation, conversely for the y plane

$$\begin{bmatrix}0\\x'\end{bmatrix} = \mathbf{M}_{CF}^{1/2FDF} = \begin{bmatrix} 1-D/f_1 & L_{rf}/2\left(1-D/f_1\right)+D \\ -\frac{1}{f^*} & -L_{rf}/2\left(\frac{1}{f^*}\right)+\left(1+D/f_2\right)\end{bmatrix}\begin{bmatrix}x\\0\end{bmatrix}$$

$$\begin{bmatrix}y\\0\end{bmatrix} = \mathbf{M}_{CF}^{1/2FDF} = \begin{bmatrix} 1+D/f_1 & L_{rf}/2\left(1+D/f_1\right)+D \\ -\frac{1}{f^{**}} & -L_{rf}/2\left(\frac{1}{f^{**}}\right)+\left(1-D/f_2\right)\end{bmatrix}\begin{bmatrix}0\\y'\end{bmatrix}$$

$$\frac{1}{f^*} = \frac{1}{f_1} - \frac{1}{f_2} + \frac{D}{f_1 f_2}$$

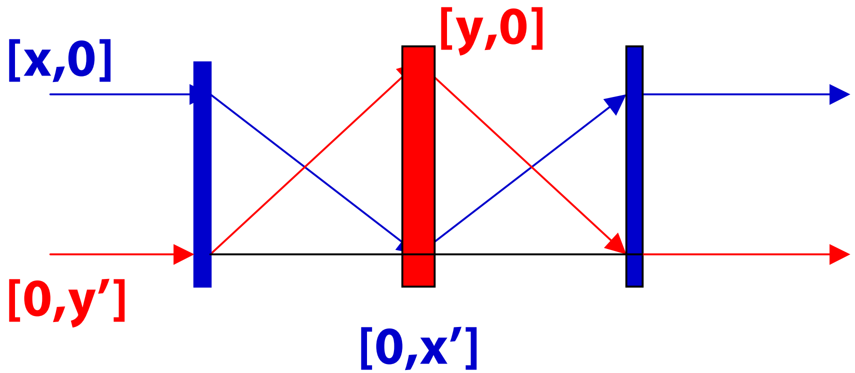
$$f_1 = D$$

$$\frac{1}{f^{**}} = -\frac{1}{f_1} + \frac{1}{f_2} + \frac{D}{f_1 f_2}; \quad f_2 = D\left(1+\frac{L_{rf}/2}{(D+L_{rf}/2)}\right)$$

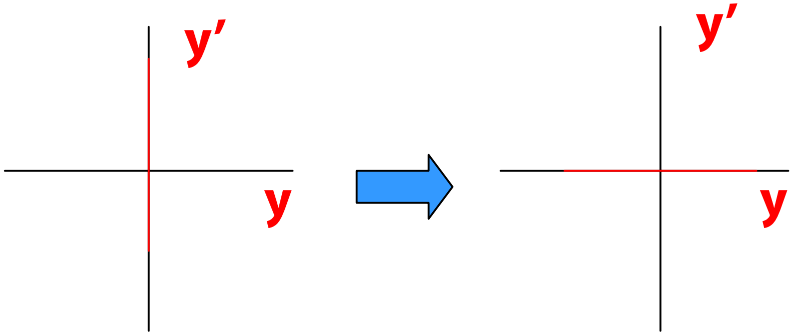
$$\begin{aligned} f_2 &\approx 2D; && L_{rf} >> D \ (\lim L \to \infty) \\ &\approx 1.4D; && D \cong L_{rf}/2 \\ &= D; && L_{rf} = 0 \end{aligned}$$

Closed orbit in the limit of stability

- These are the only closed orbits at the limits of stability:



- There is no "amplitude" transmitted, beta functions go to infinity, $\varepsilon \rightarrow 0$, phase space is a line.



FDF vs DFD split-FODO cells

- The key to understanding FDF vs DFD split FODO cells is that f^* represents the actual focal length in the horizontal plane which is a "doublet" focal length
- For FDF,

$$f_1 = D \quad and \quad \frac{1}{f^*} = \frac{1}{f_1} - \frac{1}{f_2} + \frac{D}{f_1 f_2}, \quad so \quad f^* = D$$

- In the DFD, the focal lengths reverse. f_2 = D. f_1 must be calculated specifically from the ratios of $L_{rf}/2$ to D. (For example, $L_{rf} \to \infty$ implies $f_1 \to 2D$, but the horizontal focal length f^* also $\to \infty$.) Since the ratio of D to $L_{rf}/2$ is about 2:1 (the magnet half-cell lengths are $\cong L_{rf}$), thus

$$f_1 = 1.3D, \qquad and \quad f^* = 2D$$

Final Comparison, FODO vs. split-FODO

- One can now compare the FODO (*f* = L @90 °) with the DFD and the FDF configurations, and the decrease in dispersion and momentum compaction is clear in the limit of stability

$$\eta_{CF}^{max} = \frac{f^2}{L}\vartheta_B = L\vartheta_B \cong \vartheta_B 1.5D$$

$$\eta_{CF}^{min} = \frac{f^2}{L}\vartheta_B\left(1-\frac{L}{f}\right) = 0$$

$$\eta_{DFD}^{max} = \vartheta_B 2D$$

$$\eta_{DFD}^{min} = \vartheta_B f^*\left(1 - D/f_1\right) \cong 1/4\,\eta_{DFD}^{max}$$

$$\eta_{FDF}^{max} = \vartheta_B D$$

$$\eta_{FDF}^{min} = 0$$

FODO DFD FDF

- At this point, one can resort to scaling to generalize conclusions for the focal length and can be used over the entire momentum range in the thin-lens approximation.

$$f(p) = \frac{p}{p_{\lim}} f_{\min} = \frac{p}{p_l} f_{\min}$$

10-20 GeV “Nonscaling” FFAGs: Examples

	FDF-triplet	FODO
Circumference	607m	616m
#cells	110	108
cell length	5.521m	5.704m
D-bend length	1.89m	1.314m
F-bend length	0.315m (2!)	0.390
F-D spacing	0.5 m	0.5m
Central energy	20 GeV	18 GeV
F gradient	60 T/m	60 T/m
D gradient	20 T/m	18 T/m
F strength	0.99	0.9384
D strength	0.300	0.300
Bend-field (central energy)	2 T	2.7 T
Orbit swing		
Low	-7.7	-9.8
High	0	3.8
ΔC (pathlength)	16.6	26
βxmax/βymax (10 GeV)	6.5/13.8	14.4/11.44
β(injection straight)	6.5	5.8

Analytical solutions for optimized non-scaling lattices

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- As was the case for dispersion, thin lens equations offer an analytical solution for optimized FODO and split-FODO optics and give very close starting points for lattice designs.

- Clearly, because of the focal length dependence of momentum compaction, the largest, stable phase advance should be imposed at the injection energy (130-160° across a FODO cell, or half that for the half-cell optics)

- Once practical strengths and drifts are in place, Geometric closure of reference orbits provide the remaining constraint.
  - In a completely periodic ring, geometric closure requires that the total bend angle across a period is equal for all orbits. This is also true within the period between points of reflective symmetry, or the half cell.
  - Closure can simply be imposed to find solutions for stabile lattices.

Closure and phase advance

- Geometric closure: the total bend experienced by all reference orbits must equal that of the central orbit at all momenta:

$$\vartheta_{1/2cell} = \vartheta_F^l + \vartheta_B^l = \vartheta_F^u + \vartheta_B^u + \vartheta_D^u$$

  - Where $\vartheta_F^{\,l}$ ($\vartheta_F^{\,u}$) is the kick experienced through the horizontally focusing quadrupole and $\vartheta_B^{\,l}$ ($\vartheta_B^{\,u}$) is the pure dipole bend in the vertically focusing, combined function magnet, and 0($\vartheta_D^{\,u}$) is the contribution to the total kick from the vertically-focusing gradient in this magnet for lower(upper) momentum.
  - It is assumed for simplicity that the focal length is equal to the half-cell length at low-momentum so the orbit arrives at the midplane point of the combined-function magnet and there is no contribution from the gradient. (The phase advance is so large at low momentum that $\vartheta_D^{\,u} \sim 0$ is a good approxmation.)

- The quantity $\vartheta_D{}^u$ can be expressed in terms of the horizontally-focusing quadrupole kick if the horizontal and vertical focusing strengths are set equal. This condition also turns out to be optimal for stability over the large range in momentum and phase advance. In this case

$$\vartheta_D^u = \frac{(\Delta x - L_{1/2}\vartheta_F^u)}{\Delta x}\vartheta_F^u = (1 - \frac{(L_{1/2}\vartheta_F^u)}{\Delta x})\vartheta_F^u$$

- since the crossing offset at the combined function magnet is given by $\Delta x - L_{1/2}\vartheta_F^u$, with Δx representing the reference particle position in the horizontally-focusing quadrupole

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- At this point, a practical aperture consideration can be imposed—the high and low momentum orbits are fixed at equal offsets on either side of the zero-gradient point in the horizontally-focusing quadrupole—in FODO optics, this actually symmetrizes the parabolic circumference change

- In this case the horizontally-focusing quadrupole kicks can be expressed in terms of either the high or low momentum kick using the injection/extraction p^l / p^u momentum ratio
.

$$\vartheta_B^l - \vartheta_F^l = \frac{p^l}{p^u}[\vartheta_B^l + \vartheta_F^l - \vartheta_F^l + \frac{L_{1/2}}{\Delta x}\frac{p^l}{p^u}(\vartheta_F^l)^2]$$

- $\vartheta_F{}^u$, $\vartheta_B{}^l$, and $\vartheta_B{}^u$ are positive angles (for a clockwise bend of particle trajectory) and therefore $\vartheta_F{}^l$ and $\vartheta_D{}^u$ are negative.

- This quadratic equation can now be solved for ϑ_F^l in terms of ϑ_B^l.

$$\vartheta_F^l = \frac{-1 \pm \sqrt{1 + 4\frac{L_{1/2}}{\Delta x}\left(\frac{p^l}{p^u}\right)^2\left(1 - \frac{p^l}{p^u}\right)\vartheta_B^l}}{2\left(\frac{p^l}{p^u}\right)^2 \frac{L_{1/2}}{\Delta x}}$$

- expanding square root in a Taylor series and keeping only the first two terms yields the simple equation:

$$\vartheta_F^l = \left(1 - \frac{p^l}{p^u}\right)\vartheta_B^l$$

- This result is clearly correct since as $p^l / p^u \rightarrow 0$, the kick in the horizontally focusing quadrupole must cancel the increasing kick from the dipole field as momentum is lowered. The total bend angle in this limit approaches zero.

- Geometric closure is not the only stability constraint; transverse stability in amplitude is also a requirement (closure of beta functions). Analytically this constraint can be expressed in terms of the half cell length times the kick experienced in the horizontally-focusing quadrupole for an orbit offset relative to the central orbit.

$$\Delta x = L_{1/2}\vartheta_F^l$$

Where Δx remains the offset of reference injection orbit from the central orbit. (This is a condition on the focal length of the quadrupole at low energy.)

- At this point it is useful to express equation is terms of ultimately technical choices: field strengths and magnet lengths and apertures:

$$\vartheta_F^l = \frac{0.3 l_F B_F}{p^l} = \left(1 - \frac{p^l}{p^u}\right)\vartheta_B^l = \left(1 - \frac{p^l}{p^u}\right)\frac{0.3 l_B B_B}{p^l}$$

$$giving \quad l_B = \frac{B_F l_F}{B_B\left(1 - \frac{p^l}{p^u}\right)}$$

$$\Delta x = L_{1/2}\vartheta_F^l = (l_F + l_S + l_B)\frac{0.3 l_F B_F}{p^l}$$

- Where l_B, l_F, and l_s are the lengths of the combined-function magnet, the horizontally-focusing one, and the rf drift, respectively with corresponding notation for the B fields

- Combining the two equations gives a solvable quadratic equation expressed completely in terms of field strengths, lengths, and apertures.

$$l_F = \frac{-l_S + \sqrt{l_S^2 + \frac{4 p_l \Delta x}{0.3 B_F} a}}{2a} \qquad with \qquad a = \left(1 + \frac{B_F}{B_B(1 - p^l / p^u)}\right)$$

and l_B *is given by the previous equation*

- For example, in recent designs, B_F and B_B have been chosen to be 6T and 4T, respectively, to keep pole-tip field strengths of the magnets under 7T to retain standard superconducting technology. Once these fields are chosen along with a magnet aperture, an "initial" optimal lattice is completely determined. Similar equations have been developed for the split-FODO.

Solving the FDF modified FODO

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- One can exercise the same equations and find identical results for the FDF modified-FODO case.

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